

# GLOBAL IN TIME CLASSICAL SOLUTIONS TO THE 3D QUASI-GEOSTROPHIC SYSTEM FOR LARGE INITIAL DATA

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**ABSTRACT.** In this paper, the authors show the existence of global in time classical solutions to the 3D quasi-geostrophic system with Ekman pumping for any smooth initial value (possibly large). This system couples an inviscid transport equation in  $\mathbb{R}_+^3$  with an equation on the boundary satisfied by the trace. The proof combines the De Giorgi regularization effect on the boundary  $z = 0$ —similar to the so called surface quasi-geostrophic equation—with Beale-Kato-Majda techniques to propagate regularity for  $z > 0$ . A potential theory argument is used to strengthen the regularization effect on the trace up to the Besov space  $\dot{B}_{\infty,\infty}^1$ .

## 1. INTRODUCTION

We consider the 3D quasi-geostrophic system (QG), which can be stated as the following set of equations imposed upon the stream function  $\Psi : [0, \infty) \times \mathbb{R}_+^3 \rightarrow \mathbb{R}$ :

$$\begin{cases} \partial_t(\Delta\Psi) + \overline{\nabla}^\perp\Psi \cdot \overline{\nabla}(\Delta\Psi) = 0 & t > 0, \quad z > 0, \quad x = (x_1, x_2) \in \mathbb{R}^2 \\ \partial_t(\partial_\nu\Psi) + \overline{\nabla}^\perp\Psi \cdot \overline{\nabla}(\partial_\nu\Psi) = \overline{\Delta}\Psi & t > 0, \quad z = 0, \quad x = (x_1, x_2) \in \mathbb{R}^2 \\ \Psi(0, z, x) = \Psi_0(z, x) & t = 0, \quad z \geq 0, \quad x = (x_1, x_2) \in \mathbb{R}^2 \end{cases} \quad (\text{QG}) .$$

As a convention, we choose the vertical component to be the first component of any vector in  $\mathbb{R}_+^3$ . Here we employ the following notation:

$$\overline{\nabla}\Psi = (0, \partial_{x_1}\Psi, \partial_{x_2}\Psi),$$

and

$$\overline{\Delta}\Psi = \partial_{x_1x_1}\Psi + \partial_{x_2x_2}\Psi.$$

The velocity field for the stratified flow is given by

$$\overline{\nabla}^\perp\Psi = (0, -\partial_{x_2}\Psi, \partial_{x_1}\Psi).$$

The 3D quasi-geostrophic system is a widely used model in oceanography and meteorology to describe large-scale oceanic and atmospheric circulation. The system includes two coupled equations. First, beginning with Navier-Stokes and accounting for the rotation of the Earth, one derives a transport equation on the vorticity. Second, a careful analysis of the Ekman layers near the boundary produces an equation which  $\partial_\nu\Psi$  satisfies. Chemin [8] considered the convergence in the limit of solutions to the primitive equations to a solution of the quasi-geostrophic equation. In addition, rigorous derivations were carried out by Beale and Bourgeois [5] in the absence of boundary layers and Desjardins and Grenier

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[16] with the inclusion of the boundary layers. Much of the difficulty in the analysis in fact stems from the boundary layers. Taking advantage of the viscous term on the boundary, Desjardins and Grenier [16] constructed global weak solutions. Recently, global weak solutions were constructed in the inviscid case [25]. Much recent work has also been focused on a simplified model first studied by Constantin, Majda, and Tabak [12] and known as the surface quasigeostrophic equation (SQG). There are different variants of SQG depending on the strength of the diffusive term. In the critical case, global regularity has been obtained by several different authors, each utilizing different techniques; see Kiselev, Nazarov, and Volberg [22], [7], Kiselev and Nazarov [21], and Constantin and Vicol [13]. Many authors have also emphasized the connection between critical SQG and 3D Navier-Stokes and have used versions of SQG, especially the inviscid one, as toy models for 3D fluid equations (see Constantin [11] and Held, Garner, Pierrehumbert, and Swanson [19]).

This paper is dedicated to a proof of the following well-posedness result for (QG).

**Theorem 1.1.** *Let the initial data  $\nabla\Psi_0 \in H^s(\mathbb{R}_+^3)$  for some  $s \geq 3$ . Then there exists a unique classical solution  $\Psi$  to (QG) satisfying the following: for all  $T > 0$ , there exists  $C(T, s)$  such that for all  $t \leq T$ ,  $\|\nabla\Psi(t, \cdot)\|_{H^s(\mathbb{R}_+^3)} \leq C(T, s)$ . In addition, if the initial data  $\nabla\Psi_0 \in H^s(\mathbb{R}_+^3)$  for all  $s$ , then for all  $T$ ,  $\Psi \in C^\infty([0, T] \times \mathbb{R}_+^3)$ .*

The bulk of the proof is centered around verifying a version of the Beale-Kato-Majda criterion from [3]. The idea is to first decompose the solution  $\Psi = \Psi_1 + \Psi_2$  into two components as follows:

$$\begin{cases} \Delta\Psi_1 = 0 \\ \partial_\nu\Psi_1 = \partial_\nu\Psi \end{cases} \quad \begin{cases} \Delta\Psi_2 = \Delta\Psi \\ \partial_\nu\Psi_2 = 0. \end{cases}$$

Intuitively,  $\Psi_1$  is the problematic term since it contains the boundary condition. We will find that  $\partial_\nu\Psi_1$  satisfies an equation resembling critical 2D SQG, with an adjustment to the drift term and a forcing term appearing from the presence of  $\Psi_2$ . To show that  $\partial_\nu\Psi_1$  is Hölder continuous, we utilize the De Giorgi technique following [7] and [27]. We then improve the regularity using Littlewood-Paley techniques to bootstrap (see Constantin and Wu [14] and Dong and Pavlović [17]) and potential theory to handle the forcing term coming from  $\Psi_2$ . In order to then show global well-posedness, one generally requires Lipschitz regularity or a suitable substitute on the velocity  $\overline{\nabla}^\perp\Psi$ . Due to the fact that  $\partial_x\Psi_1, \partial_y\Psi_1$  are related to  $\partial_z\Psi_1$  via the Riesz transforms and the fact that  $\overline{\nabla}\Psi_2$  is not even Lipschitz, the Besov version of Lipschitz regularity must suffice. In the literature, this space is referred to as  $\dot{B}_{\infty,\infty}^1$ , or the Zygmund class. The texts of Stein [26] and Grafakos [18] include thorough expositions of the essential theory, while Chemin [9] and Bahouri, Chemin, and Danchin [2] have detailed the application of the Zygmund class to the study of wide classes of PDE's, particularly the incompressible Euler equations. For us, the most useful property of Besov spaces will be an inequality which controls the  $L^\infty$  norm by the  $\dot{B}_{\infty,\infty}^0$  Besov norm, a lower Sobolev norm, and a logarithm of a higher Sobolev norm. From there, we can prove propagation of regularity.

The first section of the paper sets the notation and recalls some necessary results. The second section contains the proof of the  $C^\alpha$  regularity on  $\partial_\nu\Psi_1$ . In the third section, we bootstrap the regularity of  $\partial_\nu\Psi_1$  (and therefore  $\nabla\Psi_1$ ) up to  $\dot{B}_{\infty,\infty}^1$ . In the last section, we show the propagation of regularity. The appendix provides sketches of certain results for which the essential ideas can be found in Bahouri, Chemin, and Danchin [2], Chemin [9], and Constantin and Wu [14]; we record them here for the sake of completeness and readability.

## 2. NOTATION AND PRELIMINARIES

We use the notation  $L^p(\mathbb{R}^n)$  for the Lebesgue spaces. We denote the usual Hilbert Sobolev spaces (for fractional and integer  $s$ ) by  $H^s(\mathbb{R}^n)$ . The homogeneous Sobolev spaces are denoted  $\dot{H}^s(\mathbb{R}^n)$  and are defined as the space of functions  $f$  such that  $(-\Delta)^{\frac{s}{2}}f \in L^2$ . We use the notation  $\nabla^s f$  to denote the collection of all partial derivatives of order  $s \in \mathbb{N}$ .

In this paper, we consider functions defined on  $\mathbb{R}^2$  or  $\mathbb{R}_+^3 = [0, \infty) \times \mathbb{R}^2$ . It will be convenient to keep track of when functions are being differentiated in  $x$  and  $y$  only. For that reason, and also to emphasize when we are considering functions defined on  $\mathbb{R}^2$ , we employ the following notations.

**Definition 2.1.** *Let  $f$  be a real-valued function defined on  $\mathbb{R}_+^3$ . Put  $\overline{\Delta}f = \partial_{x_1x_1}f + \partial_{x_2x_2}f$  and  $\overline{\nabla}f = (0, \partial_{x_1}f, \partial_{x_2}f)$ . Let  $((-\overline{\Delta})^\alpha f)^\wedge(z, \xi) = \hat{f}(z, \xi) \cdot |\xi|^{2\alpha}$ , where the Fourier transform is being taken in  $x$  only for each fixed  $z$  (ignoring constants coming from the Fourier transform). For a partial differential operator with multi-index  $\alpha = (\alpha_1, \alpha_2)$ ,  $\overline{D}^\alpha f$  denotes differentiation in the flat variables  $(x_1, x_2)$ . When  $f$  is only defined on  $\mathbb{R}^2$ , we will use the above symbols to denote the usual differential operators.*

We recall the well known fact that the characteristic function  $\chi_E$  of a bounded, Lebesgue measurable set  $E$  belongs to  $H^s$  if and only if  $s < \frac{1}{2}$  (see Bourgain, Brezis, and Mironescu [4] for a detailed discussion). The following is a corollary which will be necessary to prove the decrease in oscillation in the De Giorgi argument.

**Proposition 2.1.** *Let  $\phi$  be a radially symmetric and decreasing,  $C^\infty$  bump function such that  $0 \leq \phi(x) \leq 1$  for all  $x$ ,  $\phi = 1$  on  $B_1(0)$ , and  $\text{supp } \phi \subset B_2(0)$ . Let  $E \subset \text{supp } \phi$ . Then  $\chi_E \cdot \phi$  belongs to  $H^{\frac{1}{2}}(\text{supp } \phi)$  if and only if  $|E| = 0$  or  $|E| = |\text{supp } \phi|$ .*

Lipschitz spaces and their variants will be referred to frequently throughout.

**Definition 2.2.** (1) For  $\alpha \in (0, 1)$ , let  $C^\alpha = \{f : \|f\|_{C^\alpha} < \infty\}$ , where

$$\|f\|_{C^\alpha} = \|f\|_{L^\infty} + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha}.$$

Also, the homogeneous space  $\dot{C}^\alpha$  is defined as

$$\{f : \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha} < \infty\}.$$

(2) Let  $\text{Lip} = \{f : \|f\|_{\text{Lip}} < \infty\}$ , where

$$\|f\|_{\text{Lip}} = \|f\|_{L^\infty} + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|}.$$

(3) Let the space of log-Lipschitz functions  $\text{LL} = \{f : \|f\|_{\text{LL}} < \infty\}$ , where

$$\|f\|_{\text{LL}} = \|f\|_{L^\infty} + \sup_{|x-y| < 1, x \neq y} \frac{|f(x) - f(y)|}{|x - y|(1 - \log(|x - y|))}.$$

Let us now recall the classical Littlewood-Paley operators and the relevant function spaces, as well as some equivalences. Let  $\mathcal{S}(\mathbb{R}^n)$  denote the Schwarz class of rapidly decaying smooth functions, and  $\mathcal{S}'(\mathbb{R}^n)$  the dual space of tempered distributions. Letting  $\mathcal{P}$  denote the space of polynomials, we construct the space  $\mathcal{S}'/\mathcal{P}$ , i.e., tempered distributions modulo polynomials.

We employ the standard dyadic decomposition of  $\mathbb{R}^n$ , specifically a sequence of smooth functions  $\{\Phi_j\}_{j \in \mathbb{Z}}$  such that

$$\text{supp } \hat{\Phi}_j \subset \{\xi \in \mathbb{R}^n : |\xi| \in (2^{j-1}, 2^{j+1})\}$$

and

$$\sum_{j \in \mathbb{Z}} \hat{\Phi}_j(\xi) = \begin{cases} 0 & \text{if } \xi = 0 \\ 1 & \text{if } \xi \in \mathbb{R}^n \setminus \{0\}. \end{cases}$$

For  $f \in \mathcal{S}'/\mathcal{P}$  and  $j \in \mathbb{Z}$ , we define  $\Delta_j f = \Phi_j * f$ .

**Definition 2.3.** For  $s \in \mathbb{R}$  and  $1 \leq p, q \leq \infty$ , the space  $\dot{B}_{p,q}^s$  is defined as

$$\{f \in \mathcal{S}'/\mathcal{P} : \|f\|_{\dot{B}_{p,q}^s} < \infty\}$$

where the homogeneous Besov norm is defined as the  $l^p$  norm of the doubly-infinite sequence of Littlewood-Paley projections:

$$\|f\|_{\dot{B}_{p,q}^s} = \|\{2^{js} \|\Delta_j f\|_{L^q}\}_{j \in \mathbb{Z}}\|_{l^p}.$$

In nearly every usage throughout the paper, the Littlewood-Paley projections and the accompanying Besov norms are in  $x = (x_1, x_2)$  only; for clarity and emphasis we will use the notation  $\dot{B}_{p,q}^s(\mathbb{R}^2)$ .

We record the following Bernstein's inequality (see [2]), which describes the action of Fourier multipliers on functions comprised of frequencies confined to an annulus.

**Proposition 2.2.** Let  $\mathcal{C}$  be an annulus in  $\mathbb{R}^d$ ,  $m \in \mathbb{R}$ , and  $k = 2\lfloor 1 + \frac{d}{2} \rfloor$ . Let  $\sigma$  be a  $k$ -times differentiable function on  $\mathbb{R}^d \setminus \{0\}$  such that for any  $\alpha \in \mathbb{N}^d$  with  $|\alpha| \leq k$ , there exists a constant  $C_\alpha$  such that

$$\forall \xi \in \mathbb{R}^d, |D^\alpha \sigma(\xi)| \leq C_\alpha |\xi|^{m-|\alpha|}.$$

There exists a constant  $C$ , depending only on the constants  $C_\alpha$ , such that for any  $p \in [1, \infty]$  and any  $\lambda > 0$ , we have, for any function  $u$  in  $L^p$  with Fourier transform supported in  $\lambda\mathcal{C}$ ,

$$\|(\sigma(\xi)\hat{u}(\xi))^\vee\|_{L^p} \leq C\lambda^m \|u\|_{L^p}.$$

We collect two embedding theorems and several corollaries of Bernstein's inequality in the following proposition.

**Proposition 2.3.** Let  $s \in \mathbb{R}$ ,  $p, q \in [1, \infty]$ .

- (1) Let  $\mathcal{R}_j$  denote the  $j^{\text{th}}$  Riesz transform with Fourier multiplier  $\frac{i\xi_j}{|\xi|}$ . Then  $\mathcal{R}_j$  is a bounded linear operator from  $\dot{B}_{p,q}^s$  to itself.
- (2) Let  $\alpha$  be a multi-index. Then the partial differential operator  $D^\alpha$  is bounded from  $\dot{B}_{p,q}^s$  to  $\dot{B}_{p,q}^{s-|\alpha|}$ .
- (3) Given  $\alpha \in \mathbb{R}$ , the operator  $(-\Delta)^\alpha$  is bounded from  $\dot{B}_{p,q}^s$  to  $\dot{B}_{p,q}^{s-2\alpha}$ .
- (4) If  $1 \leq q_1 \leq q_2 \leq \infty$ , then  $\dot{B}_{p,q_1}^s \subset \dot{B}_{p,q_2}^s$ .
- (5) If  $1 \leq p_1 \leq p_2 \leq \infty$  and  $s_1 = s_2 + n(\frac{1}{p_1} - \frac{1}{p_2})$ , then  $\dot{B}_{p_1,q}^{s_1}(\mathbb{R}^n) \subset \dot{B}_{p_2,q}^{s_2}(\mathbb{R}^n)$ .

We collect several facts concerning the Besov spaces  $\dot{B}_{\infty,\infty}^s$ . For a more detailed discussion as well as proofs, see [18].

**Proposition 2.4.** (1) The space  $\dot{B}_{\infty,\infty}^1$  can be characterized as the space of functions such that

$$\|f\|_{\dot{B}_{\infty,\infty}^1} = \sup_{x,y \in \mathbb{R}^n, y \neq 0} \frac{|f(x+y) + f(x-y) - 2f(x)|}{|y|} < \infty$$

with equivalence in norm holding between the difference quotient and Littlewood-Paley characterizations.

- (2) For non-integer values of  $s$ , the spaces  $\dot{B}_{\infty,\infty}^s$  and  $\dot{C}^s$  are equivalent, with an equivalence in norm (which is not uniform in  $s$ ).
- (3) For positive, non-integer  $s$ , the restriction of any function  $f \in \dot{B}_{\infty,\infty}^s(\mathbb{R}^n)$  to any  $k$ -dimensional affine subset produces a function in  $\dot{B}_{\infty,\infty}^s(\mathbb{R}^k)$ .

We shall need to control the  $L^\infty$  norm of a function by the  $\dot{B}_{\infty,\infty}^0$  Besov norm and some Sobolev norms. The following inequality will suit our purposes; the proof follows that of Proposition 2.104 in [2], and we include it in the appendix. See also [9] for the same result.

**Proposition 2.5.** There exists a constant  $C$  such that for any  $h : \mathbb{R}^2 \rightarrow \mathbb{R}$ ,

$$\|h\|_{L^\infty} \leq C\|h\|_{H^{-1}} + C\|h\|_{\dot{B}_{\infty,\infty}^0} \left( 1 + \log \frac{\|h\|_{\dot{H}^{\frac{3}{2}}}}{\|h\|_{\dot{B}_{\infty,\infty}^0}} \right).$$

In order to prove propagation of regularity, we shall use the classical commutator estimate whose proof may be found in Klainerman and Majda [23]. In our case, the control of  $\|\nabla f\|_{L^\infty}, \|g\|_{L^\infty}$  will come from the Besov regularity of  $f$  and  $g$  and Proposition 2.5.

**Proposition 2.6.** Assume  $f, g \in H^s(\mathbb{R}^n)$ . Then for any multi-index  $\alpha$  with  $|\alpha| = s$ , we have

$$\|D^\alpha(fg) - fD^\alpha g\|_{L^2} \leq C(s) (\|\nabla f\|_{L^\infty} \|\nabla^{(s-1)} g\|_{L^2} + \|g\|_{L^\infty} \|\nabla^s f\|_{L^2}).$$

Constantin and Wu [14] and Dong and Pavlović [17] considered solutions to the supercritical surface quasi-geostrophic equation in  $\mathbb{R}^2$  and gave conditions under which a weak solution is in fact a classical solution. The proof in [14] relies on a paraproduct argument which we have adapted to our setting. Let us emphasize that the proof is essentially identical; we give an outline in the appendix for the sake of completeness.

**Proposition 2.7.** Let  $\theta : \mathbb{R}^2 \rightarrow \mathbb{R}$  be a Leray-Hopf weak solution to

$$\partial_t \theta + u \cdot \nabla \theta + (-\Delta)^{\frac{1}{2}} \theta = 0.$$

Then if  $\operatorname{div} u = 0$  and

$$u, \theta \in L^\infty([t_0, t]; C^\delta \cap L^2)$$

we have

$$\theta \in L^\infty([t_0, t]; C^{2\delta-\epsilon})$$

for every  $\epsilon > 0$ .

We will require the following lemmas concerning BMO functions to carry out the De Giorgi argument. Here we use BMO to refer to the space of functions with bounded mean oscillation equipped with the usual norm. The first two lemmas are well-known properties of functions belonging to BMO (see [18]). The third follows from the John-Nirenberg inequality. The fourth follows from the third in conjunction with a generalization of the Cauchy-Lipschitz theorem for  $L^1(\text{LL})$  vector fields (see Theorem 3.7 in Chapter 3 of [2]). Integrals with a dash through the center are average values.

**Proposition 2.8.** (1) Let  $Q$  denote any cube in  $\mathbb{R}^n$ . For all  $0 < p < \infty$ , there exists a finite constant  $B_{p,n}$  such that

$$\sup_Q \left( \oint_Q \left| f - \oint_Q f \right|^p \right)^{\frac{1}{p}} \leq B_{p,n} \|f\|_{\text{BMO}}.$$

(2) Let  $B_1, B_2$  be two balls in  $\mathbb{R}^n$  such that there exists  $A$  such that

$$A^{-1} \text{diam}(B_2) \leq \text{diam}(B_1) \leq A \text{diam}(B_2)$$

and

$$\text{dist}(B_1, B_2) \leq A \text{diam}(B_1).$$

Then there exists a constant  $C(A)$  such that for any  $u \in \text{BMO}$

$$\left| \oint_{B_1} u - \oint_{B_2} u \right| \leq C(A) \|u\|_{\text{BMO}}.$$

(3) Let  $u \in \text{BMO}$  and satisfy

$$\sup_{x \in \mathbb{R}^n} \oint_{B_1(x)} u(y) dy < \infty$$

and define

$$f(x) = \oint_{B_1(x)} u(y) dy.$$

Then  $f(x)$  is log-Lipschitz (LL) in  $x$ .

(4) Let  $u(t, x) : [-2, 0] \times \mathbb{R}^2 \rightarrow \mathbb{R}$  belong to  $L^\infty([-2, 0]; \text{BMO}(\mathbb{R}^2)) \cap L^\infty([-2, 0]; L^2(\mathbb{R}^2))$ . Then the following ordinary differential equation has a unique Lipschitz solution which satisfies the ODE almost everywhere in time.

$$\begin{cases} \dot{\Gamma}(t) = \oint_{B_1(\Gamma(t))} u(t, y) dy \\ \Gamma(0) = 0 \end{cases}$$

In [25], the authors prove the existence of weak solutions to the inviscid quasi-geostrophic system. The proof reformulates the system into a transport equation on  $\nabla \Psi$  and relies on the following orthogonal decomposition of  $L^2$  vector fields. Given an  $L^2$  vector field  $u$ , we can decompose  $u$  as  $u = \mathbb{P}_\nabla u + \mathbb{P}_{\text{curl}} u$ . Here  $\mathbb{P}_\nabla u = \nabla v$  for some scalar function  $v$ , and  $\partial_\nu u = \partial_\nu \mathbb{P}_\nabla u$ . Furthermore,  $\mathbb{P}_{\text{curl}} u = \text{curl}(w)$  for some  $L^2$  vector field  $w$ , and  $\partial_\nu \mathbb{P}_{\text{curl}} u = 0$ . Note that the operator  $\mathbb{P}_\nabla$  commutes with  $\overline{D}^\alpha$  but not  $D^\alpha$ .

**Proposition 2.9.** Let  $\Psi$  be a smooth solution to (QG). Then if  $F$  solves the Neumann problem with  $\Delta F = 0$  and  $\partial_\nu F = \overline{\Delta} \Psi|_{z=0}$ ,  $\nabla \Psi$  satisfies the following equation:

$$\partial_t(\nabla \Psi) + \mathbb{P}_\nabla(\overline{\nabla}^\perp \Psi \cdot \overline{\nabla}(\nabla \Psi)) = \nabla F.$$

Finally, we state a local existence theorem and the *a priori* estimates satisfied by  $\Psi$ ,  $\Psi_1$ , and  $\Psi_2$ . For a proof of the local existence theorem, one can employ the standard semigroup approach found in Kato [20].

**Proposition 2.10.** Given smooth initial data  $\nabla \Psi_0$  for (QG), there exists a time interval  $[0, T]$ , where  $T$  depends only on the size of  $\|\nabla \Psi_0\|_{H^3(\mathbb{R}_+^3)}$ , such that (QG) has a smooth solution.

**Proposition 2.11.** *Given initial data  $\nabla \Psi_0 \in H^3$ , then for any finite time interval  $[0, T]$ ,  $\nabla \Psi$ ,  $\nabla \Psi_1$ , and  $\nabla \Psi_2$  satisfy the following:*

- (1)  $\nabla \Psi \in L^\infty([0, T]; H^{\frac{1}{2}}(\mathbb{R}_+^3)) \cap L^2([0, T]; H^1(\mathbb{R}_+^3))$
- (2)  $\nabla \Psi_1 \in L^\infty([0, T]; H^{\frac{1}{2}}(\mathbb{R}_+^3)) \cap L^2([0, T]; H^1(\mathbb{R}_+^3))$ , and  $\nabla \Psi_1|_{z=0} \in L^\infty([0, T]; L^2(\mathbb{R}^2)) \cap L^2([0, T]; H^{\frac{1}{2}}(\mathbb{R}^2))$
- (3) For any  $p \in (1, \infty)$ ,  $\nabla \Psi_2 \in L^\infty([0, T]; W^{1,p}(\mathbb{R}_+^3))$ . For  $p = \infty$ ,  $z_0 \geq 0$ , and  $\alpha \in (0, 1)$ ,  $\nabla \Psi_2|_{z=z_0} \in L^\infty([0, T]; \dot{B}_{\infty,\infty}^1(\mathbb{R}^2) \cap C^\alpha(\mathbb{R}^2))$ .

*Proof.* Part (1) and the first assertions in (2) and (3) follow from the classical construction of weak solutions. For the Besov and Hölder regularity of  $\Psi_2$ , notice that by Sobolev embedding and the incompressibility of the flow,  $\Delta \Psi_2(t, \cdot) \in L^p(\mathbb{R}_+^3)$  for all  $p \in [1, \infty]$ . Utilizing the fact that  $\partial_\nu \Psi_2 = 0$  and extending  $\Psi_2$  to all of  $\mathbb{R}^3$  by defining  $\tilde{\Psi}_2(z, x) = \Psi_2(|z|, x)$ , we have that  $\Delta \tilde{\Psi}_2 \in L^p(\mathbb{R}^3)$  for all  $p \in [1, \infty]$ . Examining the Littlewood-Paley projections gives immediately that  $\tilde{\Psi}_2 \in \dot{B}_{\infty,\infty}^2(\mathbb{R}_+^3)$ . The regularity on planes  $z = z_0$  in (2) and (3) follows from taking a trace, applying the Besov embedding theorem, and using the equivalences between Besov and Hölder spaces.  $\square$

Finally, let us remark that constants  $C$  may change from line to line; if we wish to keep track of dependencies, we will write  $C(\cdot)$ .

### 3. HÖLDER REGULARITY ON THE BOUNDARY

Let us examine  $\partial_\nu \Psi_1 = \partial_\nu \Psi$ . We have that for any  $T$ ,  $\partial_\nu \Psi_1$  satisfies the equation

$$\partial_t(\partial_\nu \Psi_1) + \overline{\nabla}^\perp \Psi|_{z=0} \cdot \overline{\nabla}(\partial_\nu \Psi_1) + (-\overline{\Delta})^{\frac{1}{2}}(\partial_\nu \Psi_1) = \overline{\Delta} \Psi_2|_{z=0}$$

on  $[0, T] \times \mathbb{R}^2$ . Therefore, we shall study the regularity of solutions to an equation of the type

$$\partial_t \theta + u \cdot \overline{\nabla} \theta + (-\overline{\Delta})^{\frac{1}{2}} \theta = f.$$

We begin with the first De Giorgi lemma, which shows that for any  $T$ ,  $\theta \in L^\infty([0, T] \times \mathbb{R}^2)$ .

**Lemma 3.1** (Rough  $L^\infty$  bound). *Let  $\theta : [-2, 0] \times \mathbb{R}^2 \rightarrow \mathbb{R}$  be a solution to*

$$\partial_t \theta + u \cdot \overline{\nabla} \theta + (-\overline{\Delta})^{\frac{1}{2}} \theta = f.$$

*with  $\theta, u \in L^\infty([-2, 0]; L^2(\mathbb{R}^2)) \cap L^2([-2, 0]; H^{\frac{1}{2}}(\mathbb{R}^2))$ ,  $(-\overline{\Delta})^{-\frac{1}{4}} f \in L^\infty([-2, 0]; C^{\frac{1}{2}}(\mathbb{R}^2))$ , and  $\operatorname{div} u = 0$ . Then there exists  $M$  depending on  $f$  and  $\theta$  such that  $\theta(t, x) \leq M$  for  $(t, x) \in [-1, 0] \times \mathbb{R}^2$ .*

*Proof.* The main tool in showing the  $L^\infty$  bound is an energy inequality, which we now derive. Fix a constant  $c$ , and define  $\theta_c := (\theta - c)_+$ . Multiplying the equation by  $\theta_c$ , integrating in space, and using that the drift is divergence-free, we obtain

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} \theta_c^2(t, x) dx + \int_{\mathbb{R}^2} \theta_c(t, x) (-\overline{\Delta})^{\frac{1}{2}} \theta(t, x) dx = \int_{\mathbb{R}^2} f(t, x) \theta_c(t, x) dx.$$

Making use of a pointwise estimate of Córdoba and Córdoba [15], we have that

$$(3.1) \quad \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} \theta_c^2(t, x) dx + \int_{\mathbb{R}^2} |(-\overline{\Delta})^{\frac{1}{4}} \theta_c(t, x)|^2 dx \leq \int_{\mathbb{R}^2} f(t, x) \theta_c(t, x) dx.$$

We must estimate the right hand side in order to proceed. Put  $h(t, x) = (-\overline{\Delta})^{-\frac{1}{4}} f(t, x)$ ; we then have

$$\begin{aligned}
I &= \int_{\mathbb{R}^2} f(t, x) \theta_c(t, x) dx = \int_{\mathbb{R}^2} (-\overline{\Delta})^{\frac{1}{8}} h(t, x) (-\overline{\Delta})^{\frac{1}{8}} \theta_c(t, x) dx \\
&\leq \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{|h(t, x) - h(t, y)| |\theta_c(t, x) - \theta_c(t, y)|}{|x - y|^{\frac{5}{2}}} dx dy \\
&= \iint_{|x-y| \leq 1} \frac{|h(t, x) - h(t, y)| |\theta_c(t, x) - \theta_c(t, y)|}{|x - y|^{\frac{5}{2}}} dx dy \\
&\quad + \iint_{|x-y| > 1} \frac{|h(t, x) - h(t, y)| |\theta_c(t, x) - \theta_c(t, y)|}{|x - y|^{\frac{5}{2}}} dx dy \\
&= I_1 + I_2.
\end{aligned}$$

We begin by estimating  $I_2$ . Since  $h(t, x) \in L^\infty(C^{\frac{1}{2}})$ , we have that

$$\begin{aligned}
I_2 &\leq \iint_{|x-y| > 1} \frac{C(f) |\theta_c(t, x) - \theta_c(t, y)|}{|x - y|^{\frac{5}{2}}} dx dy \\
&\leq 4C(f) \iint_{|x-y| > 1} \frac{\theta_c(t, x)}{|x - y|^{\frac{5}{2}}} dx dy \\
&\leq C(f) \int_1^\infty r^{-\frac{3}{2}} dr \int_{\mathbb{R}^2} \theta_c(t, x) dx \\
&\leq C(f) \int_{\mathbb{R}^2} \theta_c(t, x) dx
\end{aligned}$$

We must now estimate  $I_1$ . Using the symmetry in  $x$  and  $y$  and the fact that

$$|\theta_c(t, x) - \theta_c(t, y)| \leq (\mathcal{X}_{\{\theta_c(t, x) > 0\}} + \mathcal{X}_{\{\theta_c(t, y) > 0\}}) |\theta_c(t, x) - \theta_c(t, y)|$$

we have

$$I_1 \leq 2 \iint_{|x-y| \leq 1} \mathcal{X}_{\{\theta_c(x) > 0\}} \frac{|h(t, x) - h(t, y)| |\theta_c(t, x) - \theta_c(t, y)|}{|x - y|^{\frac{5}{2}}} dx dy.$$

By Cauchy's inequality for some parameter  $a$  to be determined, we have

$$\begin{aligned}
I_1 &\leq a \iint_{|x-y| \leq 1} \mathcal{X}_{\{\theta_c(x) > 0\}} \frac{|h(t, x) - h(t, y)|^2}{|x - y|^2} dx dy \\
&\quad + \frac{1}{a} \iint_{|x-y| \leq 1} \frac{|\theta_c(t, x) - \theta_c(t, y)|^2}{|x - y|^3} dx dy.
\end{aligned}$$

Using the  $L^\infty(C^{\frac{1}{2}})$  regularity of  $h$  and choosing  $a$  large enough, we have that

$$\begin{aligned}
&a \iint_{|x-y| \leq 1} \mathcal{X}_{\{\theta_c(t, x) > 0\}} \frac{|h(t, x) - h(t, y)|^2}{|x - y|^2} dx dy \\
&\leq a \iint_{|x-y| \leq 1} \mathcal{X}_{\{\theta_c(t, x) > 0\}} \frac{\|h(t, x)\|_{L^\infty(C^{\frac{1}{2}})}^2 |x - y|}{|x - y|^2} dx dy
\end{aligned}$$



$$\begin{aligned}
&\leq C(f) \int_0^1 dr \int_{\mathbb{R}^2} \mathcal{X}_{\{\theta_c(t,x)>0\}} dx \\
&\leq C(f) \int_{\mathbb{R}^2} \mathcal{X}_{\{\theta_c(t,x)>0\}} dx
\end{aligned}$$

and

$$\frac{1}{a} \iint_{|x-y|\leq 1} \frac{|\theta_c(t,x) - \theta_c(t,y)|^2}{|x-y|^3} dx dy \leq \frac{1}{2} \|\theta_c(t, \cdot)\|_{\dot{H}^{\frac{1}{2}}}^2.$$

Substituting the estimates for  $I_1$  and  $I_2$  into Eq. (3.1) and absorbing the second part of  $I_1$  into the left hand side, we obtain the following energy inequality:

$$(3.2) \quad \frac{d}{dt} \int_{\mathbb{R}^2} \theta_c^2(t,x) dx + \int_{\mathbb{R}^2} |(-\bar{\Delta})^{\frac{1}{4}} \theta_c(t,x)|^2 dx \leq C(f) \left( \int_{\mathbb{R}^2} \theta_c(t,x) dx + \int_{\mathbb{R}^2} \mathcal{X}_{\{\theta_c(t,x)>0\}} dx \right).$$

With the energy inequality in hand, we obtain the desired nonlinear recurrence relation on the superlevel sets of energy. Let  $M > 1$  be a constant to be chosen later, and put  $M_k = M(1 - 2^{-k})$ ,  $\theta_k = (\theta - M_k)_+$  and  $T_k = -1 - 2^{-k}$ . Define

$$E_k = \sup_{t \in [T_k, 0]} \int_{\mathbb{R}^2} \theta_k^2(t,x) dx + \int_{T_k}^0 \int_{\mathbb{R}^2} |(-\bar{\Delta})^{\frac{1}{4}} \theta_k(\tau,x)|^2 dx d\tau.$$

Choose  $s \in [T_{k-1}, T_k]$  and  $t \in [T_k, 0]$ . Integrating (3.2) from  $s$  to  $t$  yields

$$\begin{aligned}
&\int_{\mathbb{R}^2} \theta_k^2(t,x) dx + \int_s^t \int_{\mathbb{R}^2} |(-\bar{\Delta})^{\frac{1}{4}} \theta_k(\tau,x)|^2 dx d\tau \\
&\leq \int_{\mathbb{R}^2} \theta_k^2(s,x) dx + C(f) \left( \int_s^t \int_{\mathbb{R}^2} |\theta_k(\tau,x)| dx d\tau + \int_s^t \int_{\mathbb{R}^2} \mathcal{X}_{\{\theta_k(\tau,x)>0\}} dx d\tau \right).
\end{aligned}$$

Now taking the supremum on the left hand side, discarding the energy at time  $s$ , and averaging over  $s \in [T_{k-1}, T_k]$  on the right hand side, we have

$$(3.3) \quad E_k \leq C(f) 2^k \left( \int_{T_{k-1}}^0 \int_{\mathbb{R}^2} \theta_k(\tau,x) dx d\tau + \int_{T_{k-1}}^0 \int_{\mathbb{R}^2} \mathcal{X}_{\{\theta_k(\tau,x)>0\}} dx d\tau \right).$$

We must control the right-hand side of (3.3) by  $E_{k-1}$  in a nonlinear fashion. First, note that Sobolev embedding gives that  $H^{\frac{1}{2}}(\mathbb{R}^2) \subset L^4(\mathbb{R}^2)$ , and using this estimate to interpolate, we obtain

$$\|\theta_k\|_{L^3([T_k, 0] \times \mathbb{R}^2)} \leq C E_k^{\frac{1}{2}}.$$

Next, we have that if  $\theta_k > 0$ , then  $\theta_{k-1} \geq 2^{-k} M$  and  $\mathcal{X}_{\{\theta_k > 0\}} \leq \frac{2^k}{M} \theta_{k-1}$ . Therefore

$$\int_{T_{k-1}}^0 \int_{\mathbb{R}^2} \theta_k dx d\tau \leq \int_{T_{k-1}}^0 \int_{\mathbb{R}^2} \theta_k \mathcal{X}_{\{\theta_k > 0\}}^2 dx d\tau \leq \frac{4^k}{M^2} \int_{T_{k-1}}^0 \int_{\mathbb{R}^2} \theta_{k-1}^3 dx d\tau \leq \frac{4^k}{M^2} E_{k-1}^{\frac{3}{2}}$$

and

$$\int_{T_{k-1}}^0 \int_{\mathbb{R}^2} \mathcal{X}_{\{\theta_k > 0\}} dx d\tau = \int_{T_{k-1}}^0 \int_{\mathbb{R}^2} \mathcal{X}_{\{\theta_k > 0\}}^3 dx d\tau \leq \frac{8^k}{M^3} \int_{T_{k-1}}^0 \int_{\mathbb{R}^2} \theta_{k-1}^3 dx d\tau \leq \frac{8^k}{M^3} E_{k-1}^{\frac{3}{2}}.$$

Combining these estimates, we have

$$E_k \leq \frac{C(f)^k}{M^2} E_{k-1}^{\frac{3}{2}}.$$

Choosing  $M$  to be large enough makes  $E_1$  arbitrarily small, showing that  $\lim_{k \rightarrow \infty} E_k = 0$  and proving the claim.  $\square$

To accommodate the second De Giorgi lemma, we must reformulate the  $L^\infty$  bound. The nonlocality of the equation makes the zooming arguments more delicate; since the decrease in oscillation required for Hölder regularity will be nonlocal in nature, we cannot use a sharp cutoff as in Lemma 3.1. To address this, we will make use of the cutoff function

$$c(x) = (|x|^{\frac{1}{4}} - 2)_+.$$

Note that  $\|(-\overline{\Delta})^{\frac{1}{2}}c\|_{L^\infty} < \infty$ . In addition, the drift term will not completely disappear after multiplying the equation by  $\theta - c(x)$  and integrating. Since  $\partial_\nu \Psi_1$  is now bounded, we have that  $\overline{\nabla}^\perp \Psi \in \text{BMO}$ . Performing a change of variables which follows the mean value of the drift through time, the new drift term will be exponentially integrable. With this in mind, we can obtain the following sharper  $L^\infty$  bound.

**Lemma 3.2** (Fine  $L^\infty$  bound). *Let  $\theta : [-2, 0] \times \mathbb{R}^2 \rightarrow \mathbb{R}$  be a solution to*

$$\partial_t \theta + u \cdot \overline{\nabla} \theta + (-\overline{\Delta})^{\frac{1}{2}} \theta = f.$$

*such that  $\theta(t, x) \leq 1 + c(x)$  on  $[-2, 0] \times \mathbb{R}^2$ ,  $\text{div } u = 0$ . Assume that there exists some universal constant  $M$  such that*

$$\|(-\overline{\Delta})^{-\frac{1}{4}}f\|_{L^\infty(C^{\frac{1}{2}})}, \|u\|_{L^\infty(L^p)} \leq M$$

*for some  $p > 3$ . Then there exists  $\delta$  depending on  $M$  such that if*

$$|\{\theta > 0\} \cap ([-1, 0] \times B_1(0))| < \delta,$$

*then  $\theta(t, x) \leq \frac{1}{2}$  for  $(t, x) \in [-\frac{1}{2}, 0] \times B_{\frac{1}{2}}(0)$ .*

*Proof.* Let  $\gamma_k$  be a bump function compactly supported in  $B_{\frac{1}{2}+2^{-k-1}}$ , equal to  $\frac{1}{2} + 2^{-k-1}$  on  $B_{\frac{1}{2}+2^{-k-2}}$ , with  $0 \leq \gamma_k \leq \frac{1}{2} + 2^{-k-1}$  for all  $x$ , and  $\gamma_k < \gamma_l$  for  $k > l$ . We will also impose that  $|\overline{\nabla} \gamma_k| \leq C2^k$  and  $|(-\overline{\Delta})^{\frac{1}{2}}\gamma_k| \leq Ck2^k$  (we provide a short justification of this condition in the appendix after the discussion of Proposition 2.5). Define  $\theta_k := (\theta - (1 + c - \gamma_k))_+$ . We multiply the equation by  $\theta_k$  and argue as before. First we record the following estimates:

$$\begin{aligned} \int_{\mathbb{R}^2} (-\overline{\Delta})^{\frac{1}{2}} \theta \theta_k dx &\leq \int_{\mathbb{R}^2} |(-\overline{\Delta})^{\frac{1}{4}} \theta_k|^2 dx + \int_{\mathbb{R}^2} (-\overline{\Delta})^{\frac{1}{2}} (1 + c - \gamma_k) \theta_k dx \\ (3.4) \quad &\leq \int_{\mathbb{R}^2} |(-\overline{\Delta})^{\frac{1}{4}} \theta_k|^2 dx + \left( \|(-\overline{\Delta})^{\frac{1}{2}} c\|_{L^\infty} + Ck2^k \right) \int_{\mathbb{R}^2} \theta_k dx \end{aligned}$$

and

$$\begin{aligned} \int_{\mathbb{R}^2} (u \cdot \overline{\nabla} \theta) \theta_k dx &= \int_{\mathbb{R}^2} (u \cdot \overline{\nabla} \theta_k) \theta_k dx - \int_{\mathbb{R}^2} (u \cdot \overline{\nabla} (1 + c - \gamma_k)) \theta_k dx \\ &= \int_{\mathbb{R}^2} - (u \cdot \overline{\nabla} (1 + c - \gamma_k)) \theta_k dx \\ (3.5) \quad &\leq C(M, q) (\|\overline{\nabla} c\|_{L^\infty} + \|\overline{\nabla} \gamma_k\|_{L^\infty}) \left( \int_{\mathbb{R}^2} \theta_k^q dx \right)^{\frac{1}{q}}. \end{aligned}$$

after applying Hölder's inequality with  $q$  on  $\theta$  and  $\frac{q-1}{q} = p$  on  $u$ . Here  $p$  is chosen to match the regularity on  $u$  in the assumption of the lemma. In addition, we can estimate the term

$$(3.6) \quad \int_{\mathbb{R}^2} f \theta_k dx$$

in the same way as Lemma 3.1. Combining estimates for (3.4), (3.5), and (3.6), we arrive at the energy inequality

$$(3.7) \quad \frac{d}{dt} \int_{\mathbb{R}^2} \theta_k^2 dx + \int_{\mathbb{R}^2} |(-\bar{\Delta})^{\frac{1}{4}} \theta_k|^2 dx \leq C(M, q) (2^k + k2^k) \left( \int_{\mathbb{R}^2} \theta_k dx + \int_{\mathbb{R}^2} \mathcal{X}_{\{\theta_k > 0\}} dx + \left( \int_{\mathbb{R}^2} \theta_k^q dx \right)^{\frac{1}{q}} \right).$$

Now define  $T_k := -\frac{1}{2} - 2^{-k-1}$ , and put

$$E_k := \sup_{t \in [T_k, 0]} \int_{\mathbb{R}^2} \theta_k^2(t, x) dx + \int_{T_k}^0 \int_{\mathbb{R}^2} |(-\bar{\Delta})^{\frac{1}{4}} \theta_k(\tau, x)|^2 dx d\tau.$$

Integrating in time, we have that the first two terms on the right hand side can be estimated as in Lemma 3.1. Using Jensen's inequality and the fact that  $\mathcal{X}_{\{\theta_k > 0\}} \leq 2^k \theta_{k-1}$ , we can estimate the third term on the right hand side of (3.7) by

$$\begin{aligned} \int_{T_{k-1}}^0 \left( \int_{\mathbb{R}^2} \theta_k^q dx \right)^{\frac{1}{q}} d\tau &\leq \int_{T_{k-1}}^0 \left( \int_{\mathbb{R}^2} \theta_k^q \mathcal{X}_{\{\theta_k > 0\}}^{3-q} dx \right)^{\frac{1}{q}} d\tau \\ &\leq C \left( \int_{T_{k-1}}^0 \int_{\mathbb{R}^2} \theta_k^q \mathcal{X}_{\{\theta_k > 0\}}^{3-q} dx d\tau \right)^{\frac{1}{q}} \\ &\leq C \left( (2^k)^{3-q} \int_{T_{k-1}}^0 \int_{\mathbb{R}^2} \theta_{k-1}^3 dx d\tau \right)^{\frac{1}{q}} \\ &\leq C^k E_{k-1}^{\frac{3}{2q}}. \end{aligned}$$

Using the integrability assumption on  $u$ , we can now choose  $q \in (1, \frac{3}{2})$ . The nonlinear recurrence relation on  $E_k$  follows as in Lemma 3.1. Noticing that (3.7) shows that choosing  $\delta$  arbitrarily small makes  $E_0$  arbitrarily small, there exists  $\delta$  such that  $\lim_{k \rightarrow \infty} E_k = 0$ . Therefore,  $\theta_k - \gamma_k$  converges to 0 in  $L^2$  for every time  $t \in [-\frac{1}{2}, 0]$ , and  $\theta \leq \lim \gamma_k$ . Thus  $\theta \leq \frac{1}{2}$  on  $[-\frac{1}{2}, 0] \times B_{\frac{1}{2}}(0)$ .  $\square$

With the  $L^\infty$  bound in hand, we turn to the second half of the De Giorgi argument. Let us remark that the most general argument is the one found in [6]. This argument works for kernels comparable to the fractional Laplacian  $(-\bar{\Delta})^\alpha$  raised to any power  $\alpha \in (0, 1)$ . However, in the case when  $\alpha \geq \frac{1}{2}$ , since the Leray-Hopf weak solutions lie in  $H^\alpha$ , we can make use of Proposition 2.1. This allows for a compactness argument following [27]. We will employ the second approach; it should be possible to adapt the argument in [6] to equations with drift and forcing terms.

First, a parabolic version of the isoperimetric lemma will be shown, following the proof in [27]. This will then imply that  $\theta$  enjoys a geometric rate of decrease in oscillation. Let  $\phi$  be a compactly supported, radially symmetric and decreasing,  $C^\infty$  bump function such that

$0 \leq \phi(x) \leq 1$  for all  $x$ ,  $\phi = 1$  on  $B_1(0)$ , and  $\text{supp } \phi \subset B_2(0)$ . Let  $\phi_0(x) = 1 + c(x) - \phi(x)$ , and  $\phi_1(x) = 1 + c(x) - \frac{1}{2}\phi(x)$ .

**Lemma 3.3** (Isoperimetric Lemma). *Assume that there exists some universal constant  $M$  such that*

$$\|(-\overline{\Delta})^{-\frac{1}{4}}f\|_{L^\infty(C^{\frac{1}{2}})}, \|u\|_{L^\infty(L^p)} \leq M$$

for some  $p > 3$ ,  $f \in L^2([-2, 0]; H^{-1}(\mathbb{R}^2))$ , and  $\text{div } u = 0$ . Fix  $\delta$  as in Lemma 3.2. Then there exists  $\alpha, \beta > 0$  (depending on  $M$ ) such that for any solution  $\theta : [-2, 0] \times \mathbb{R}^2$  to the equation

$$\partial_t \theta + u \cdot \overline{\nabla} \theta + (-\overline{\Delta})^{\frac{1}{2}} \theta = f$$

with  $\theta(x) \leq 1 + c(x)$  the following holds. Define

$$\begin{aligned} A &= \{\theta > \frac{1}{2}\} \cap ([-1, 0] \times B_1) \\ C &= \{\theta \leq 0\} \cap ([-2, -1] \times B_1) \\ D &= \{\phi_0 < \theta \leq \phi_1\} \cap ([-2, 0] \times B_2). \end{aligned}$$

Then if  $|A| \geq \delta$ ,  $|C| \geq \beta$ , then  $|D| \geq \alpha$ .

*Proof.* Assume that the lemma is false. Then, given  $\beta$  there exists a sequence of solutions  $\theta_j$  such that  $|A_j| \geq \delta$ ,  $|C_j| \geq \beta$ ,  $|D_j| \leq \frac{1}{j}$  with  $A_j$ ,  $C_j$ , and  $D_j$  defined analogously to  $A$ ,  $C$ , and  $D$ . Put  $v_j = (\theta_j - \phi_0)_+$ . We multiply by  $v_j$  and integrate. The  $L^\infty(L^p)$  bound on  $u$  gives that

$$\begin{aligned} \int_{\mathbb{R}^2} (u \cdot \overline{\nabla} \theta_j) v_j dx &= \int_{\mathbb{R}^2} (u \cdot \overline{\nabla} v_j) v_j dx + \int_{\mathbb{R}^2} (u \cdot \overline{\nabla} (1 + c - \phi)) v_j dx \\ &= \int_{\mathbb{R}^2} (u \cdot \overline{\nabla} (1 + c - \phi)) v_j dx \\ &\leq C(M, \phi, c). \end{aligned}$$

Estimating the forcing term as in Lemma 3.1 and using the  $L^\infty$  bound on  $v_j$ , we obtain the energy inequality

$$\frac{d}{dt} \int_{\mathbb{R}^2} v_j^2 + \int_{\mathbb{R}^2} |(-\overline{\Delta})^{\frac{1}{4}} v_j|^2 \leq C(M, \phi, c).$$

Integrating from  $s$  to  $t$  in time for  $-2 < s < t < 0$  gives

$$(3.8) \quad \int_{\mathbb{R}^2} v_j^2(t) + \int_s^t \int_{\mathbb{R}^2} |(-\overline{\Delta})^{\frac{1}{4}} v_j|^2 \leq \int_{\mathbb{R}^2} v_j^2(s) + C(M, \phi, c)(t - s).$$

This implies that  $v_j$  is uniformly bounded in  $L^2(\dot{H}^{\frac{1}{2}}(B_2))$ . Also, note that

$$\partial_t(\theta_j - (1 + c - \phi)) = -\text{div}(u\theta_j) - (-\overline{\Delta})^{\frac{1}{2}}\theta_j + f.$$

We have that  $(-\overline{\Delta})^{\frac{1}{2}}\theta_j, f \in L^2(H^{-1})$ . Now since  $u, \theta_j \in L^2(H^{\frac{1}{2}})$ , Sobolev embedding shows that they each belong to  $L^2(L^4)$ . By Hölder, their product belongs to  $L^2(L^2)$ , and therefore  $\text{div}(u\theta_j) \in L^2(H^{-1})$ . Therefore we have that  $\partial_t(\theta_j - (1 + c - \phi))$  belongs to  $L^2(H^{-1})$ , as does  $\partial_t v_j$ . By the Aubin-Lions compactness lemma from [1], up to a subsequence,  $v_j$  converges in  $L^2((-2, 0); L^2(B_2))$  to a function  $v$ . Also, from (3.8), the family of functions  $\{\int_{\mathbb{R}^2} v_j^2(t)\}_{j=1}^\infty$

is uniformly Lipschitz in  $t$ , and after passing to another subsequence by Arzela-Ascoli,  $v$  satisfies the inequality

$$\int_{B_2} v^2(t) \leq \int_{B_2} v^2(s) + C(M, \phi, c)(t - s).$$

Using Tchebyshev's inequality and passing to the limit, we have that

$$|\{0 < v \leq \frac{1}{2}\phi\} \cap ([-2, 0] \times B_2)| = 0.$$

By Proposition 2.1 and the fact that  $v$  belongs to  $H^{\frac{1}{2}}$  for almost every time, we have that for almost every time  $t \in (-2, 0)$ , either  $v = 0$  or  $v > \frac{1}{2}\phi$ . Since  $\int_{\mathbb{R}^2} v^2(t)$  is Lipschitz in time by the energy inequality, we have that either  $v > \frac{1}{2}\phi$  or  $v = 0$  on  $(-2, 0) \times B_2$ . Using that  $|C_j| \geq \beta$  for every  $j$ , there must exist some times  $s \in (-2, -1)$  for which  $v = 0$ , and thus by the energy inequality  $v = 0$  on all of  $[-2, 0] \times B_2$ . However, we also have that  $|A_j| \geq \delta$  for all  $j$ , contradicting the convergence of  $v_j$  to  $v$  in  $L^2(L^2)$ .  $\square$

We turn now to the oscillation lemma. We will make use of the cutoff function  $c_\epsilon(x) = (|x|^\epsilon - 2)_+$ .

**Lemma 3.4** (Decrease in Oscillation). *Let  $\delta, \beta, \alpha, C$ , and  $M$  be as in Lemma 3.3. Then there exists  $\zeta \in (0, 1)$ ,  $\epsilon > 0$ , and  $\eta > 0$  such that for any solution  $\theta : [-2, 0] \times \mathbb{R}^2$  to*

$$\partial_t \theta + u \cdot \nabla \theta + (-\overline{\Delta})^{\frac{1}{2}} \theta = f$$

with

$$\|(-\overline{\Delta})^{-\frac{1}{4}} f\|_{L^\infty(C^{\frac{1}{2}})} \leq M\eta, \|u\|_{L^\infty(L^p)} \leq M$$

for some  $p > 3$ ,  $f \in L^2(H^{-\frac{1}{2}})$ ,  $\operatorname{div} u = 0$ ,  $|C| \geq \beta$ , and  $\theta \leq 1 + c_\epsilon$ , we have  $\theta \leq 1 - \zeta$  on  $[-\frac{1}{2}, 0] \times B_{\frac{1}{2}}$ .

*Proof.* Choose  $K$  such that  $K\alpha > |(-2, 0) \times B_2|$ , and let  $\eta = 2^{-K}$ . Put  $\theta_k = 2^k(\theta - (1 - 2^{-k}))$ . By scaling,  $\theta_k$  solves the equation

$$\partial_t \theta_k + u \cdot \nabla \theta_k + (-\overline{\Delta})^{\frac{1}{2}} \theta_k = 2^k f.$$

For  $k \leq K$ ,

$$\|2^k (-\overline{\Delta})^{-\frac{1}{4}} f\|_{L^\infty(C^{\frac{1}{2}})} \leq M.$$

Choose  $\epsilon$  to be small enough such that

$$2^K (|x|^\epsilon - 2)_+ \leq (|x|^{\frac{1}{4}} - 2)_+ = c(x)$$

for all  $x$ . Note that since  $k \leq K$  we have

$$\theta_k(x) \leq 1 + 2^K c_\epsilon(x) \leq 1 + c(x).$$

Fix  $k \leq K$  now, and suppose that

$$(3.9) \quad |\{\theta_{j+1} > 0\} \cap ([-1, 0] \times B_1)| \geq \delta.$$

for all  $j \leq k$ . This implies that

$$|\{\theta_j > \frac{1}{2}\} \cap ([-1, 0] \times B_1)| \geq \delta.$$

Since  $|\{\theta_j \leq 0\} \cap ([-2, -1] \times B_1)| \geq \beta$  for all  $j$ , we have that by Lemma 3.3,

$$|\{\phi_0 < \theta_j \leq \phi_1\} \cap ([-2, 0] \times B_2)| \geq \alpha.$$

Noticing that the sets  $\{\phi_0 < \theta_j \leq \phi_1\}$ ,  $\{\phi_0 < \theta_{j'} \leq \phi_1\}$  are disjoint for  $j \neq j'$ , we have that (3.9) cannot hold for  $k = K$  by choice of  $K$ . So there must exist  $k < K$  for which

$$|\{\theta_{k+1} > 0\} \cap ([-1, 0] \times B_1)| < \delta.$$

Then by Lemma 3.2,  $\theta_{k+1} \leq \frac{1}{2}$  in  $[-\frac{1}{2}, 0] \times B_{\frac{1}{2}}$ , and  $\theta \leq 1 - 2^{-(2+K)}$  in  $[-\frac{1}{2}, 0] \times B_{\frac{1}{2}}$ , proving the claim with  $\zeta = 2^{-(2+K)}$ .  $\square$

We have arrived at Lemma 3.5 as an easy corollary, as well as the desired  $C^\alpha$  regularity.

**Lemma 3.5.** *If  $-1 - c_\epsilon \leq \theta \leq 1 + c_\epsilon$  on  $[-2, 0] \times \mathbb{R}^2$  and the conditions of Lemma 3.4 are satisfied, then on  $[-\frac{1}{2}, 0] \times B_{\frac{1}{2}}$ ,*

$$\sup \theta - \inf \theta \leq 2 - \zeta.$$

**Lemma 3.6** (Hölder Regularity). *Let  $\partial_\nu \Psi_1 = \theta$  be a solution to*

$$\partial_t \theta + \overline{\nabla}^\perp \Psi \cdot \overline{\nabla} \theta + (-\overline{\Delta})^{\frac{1}{2}} \theta = \overline{\Delta} \Psi_2$$

*with  $\nabla \Psi$  satisfying the a priori regularity estimates in Proposition 2.11. Then given  $T > 0$ , there exists  $r$  such that  $\theta \in C^r([0, T] \times \mathbb{R}^2)$ .*

*Proof.* Let  $T$  be given. Proposition 2.10 gives that the solution is smooth up to time  $T_0$  for some  $T_0$ . Given  $(t_0, x_0) \in [T_0, T] \times \mathbb{R}^2$ , we define  $K_0 = \inf(1, \frac{t_0}{4})$ . Put  $\theta_0(t, x) = \theta(t_0 + K_0 t, x_0 + K_0 x)$ , and define  $u_0, \overline{\Delta} \Psi_{2,0}$  using the same dilation by  $K_0$  and shift by  $(t_0, x_0)$ . Then  $\theta_0$  satisfies the assumptions of Lemma 3.1, and we have that  $\theta_0 \in L^\infty([-1, 0] \times \mathbb{R}^2)$ . Since the argument is translation invariant in space and we only need to consider a set of times which is bounded from above and below, we have in fact that  $\theta \in L^\infty([0, T] \times \mathbb{R}^2)$ .

Continuing to fix  $(t_0, x_0)$  and  $K_0$  as above, we will show that  $\theta$  is Hölder continuous at  $(t_0, x_0)$ . We will inductively define a sequence of dilated functions for some factor of dilation  $K$  to be determined later. Let  $\Gamma_1(t)$  be the solution to the ODE

$$\begin{cases} \dot{\Gamma}_1(t) = \int_{B_1(\Gamma_1(t))} u_0(Kt, Ky) dy \\ \Gamma_1(0) = 0 \end{cases}$$

and put

$$\begin{aligned} \theta_1(t, x) &= \frac{\theta_0(Kt, Kx + \Gamma_1(t))}{\|\theta\|_{L^\infty}} \\ u_1(t, x) &= u_0(Kt, Kx + \Gamma_1(t)) \\ \overline{\Delta} \Psi_{2,1}(t, x) &= \overline{\Delta} \Psi_{2,0}(Kt, Kx + \Gamma_1(t)). \end{aligned}$$

For  $k > 1$ , define

$$\begin{cases} \dot{\Gamma}_{k+1}(t) = \int_{B_1(\Gamma_{k+1}(t))} u_k(Kt, Ky) dy - \int_{B_1(0)} u_k(t) \\ \Gamma_{k+1}(0) = 0 \end{cases}$$

$$\theta_{k+1}(t, x) = \frac{1}{1 - \frac{\zeta}{4}} \left( \theta_k(Kt, Kx + \Gamma_{k+1}(t)) - \frac{1}{2} \left( \sup_{[-\frac{1}{4}, 0] \times B_{\frac{1}{4}}} \theta_k + \inf_{[-\frac{1}{4}, 0] \times B_{\frac{1}{4}}} \theta_k \right) \right)$$

$$u_{k+1}(t, x) = u_k(Kt, Kx + \Gamma_{k+1}(t))$$

$$\overline{\Delta} \Psi_{2,k+1}(t, x) = \overline{\Delta} \Psi_{2,k}(Kt, Kx + \Gamma_{k+1}(t)).$$

We have that  $\theta_k$  solves the equation

$$\partial_t \theta_k + (u_k - \int_{B_1(0)} u_k) \cdot \bar{\nabla} \theta_k + (-\bar{\Delta})^{\frac{1}{2}} \theta_k = \left( \frac{K}{(1 - \frac{\zeta}{4})} \right)^k \bar{\Delta} \Psi_{2,k}.$$

Examining the assumptions of the De Giorgi lemmas (Lemma 3.2, Lemma 3.3, Lemma 3.4, and Lemma 3.5), we see that  $K$  is subject to the following constraints.

- (1)  $K$  needs to be small enough to satisfy

$$\frac{1}{1 - \frac{\zeta}{2}} c_\epsilon(Kx) < c_\epsilon(x)$$

for all  $x \geq \frac{1}{K}$ .

- (2)  $K$  should be small enough so that  $\left( \frac{K}{(1 - \frac{\zeta}{4})} \right)^k \bar{\Delta} \Psi_{2,k}$  satisfies the assumptions of the De Giorgi lemmas uniformly in  $k$ . Specifically, we need  $(-\bar{\Delta})^{-\frac{1}{4}} \left( \left( \frac{K}{(1 - \frac{\zeta}{4})} \right)^k \bar{\Delta} \Psi_{2,k} \right) \in L^\infty(C^{\frac{1}{2}})$  to have small norm. Applying  $(-\bar{\Delta})^{-\frac{1}{4}}$  divides by a factor of  $K^{\frac{k}{2}}$ , but we can choose  $K$  to be very small compared to  $(1 - \frac{\zeta}{4})$ , and by using the regularity of  $\Psi_2$  we can choose  $K$  to satisfy this constraint.
- (3) We must ensure that  $u_k - \int_{B_1(0)} u_k$  satisfies the assumptions of the De Giorgi lemmas uniformly in  $k$ . However, since the BMO norm is scale invariant, this condition is satisfied independent of  $K$ .
- (4) Notice that each successive dilation includes a change of variables which follows the new flow of the dilated drift term. At the  $k^{th}$  iteration, we will obtain a decrease in oscillation for  $\theta_k$  on the set  $[-\frac{1}{2}, 0] \times B_{\frac{1}{2}}(0)$ . Then after dilating by  $K$  and shifting according to  $\Gamma_{k+1}$ , we must ensure that  $-1 - c_\epsilon \leq \theta_{k+1} \leq 1 + c_\epsilon$ . Applying Proposition 2.8, we have that  $|\dot{\Gamma}_{k+1}| < C$  for some fixed constant  $C$ . Therefore we can choose  $K$  small enough so that zooming in by a factor of  $K$  and then shifting according to the new drift gives that  $-1 - c_\epsilon \leq \theta_{k+1} \leq 1 + c_\epsilon$ .

We choose  $K$  to satisfy the above constraints. Thus we have that  $\{\theta_k\}_{k=1}^\infty$  satisfies the assumptions of the De Giorgi lemmas uniformly in  $k$ , and we obtain a decrease in oscillation of  $1 - \frac{\zeta}{4}$  for each successive iteration. To see that  $\theta$  is Hölder continuous, put

$$U_k = \sup_{[-2, 0]} |\dot{\Gamma}_k(t)|$$

and notice that the set  $[-\frac{1}{2}, 0] \times B_{\frac{1}{2}}(\Gamma_k(t))$  contains the rectangle  $[-\frac{1}{4U_k}, 0] \times B_{\frac{1}{4}}(0)$ . By Proposition 2.8, there exists  $U$  such that  $U_k \leq U$  for all  $k$ . Putting  $D = \min(\frac{K}{4}, \frac{1}{8U})$ , we have that if  $(t, x)$  is such that

$$|(t, x) - (t_0, x_0)| \approx D^k$$

then

$$|\theta(t, x) - \theta(t_0, x_0)| \leq \left( 1 - \frac{\zeta}{4} \right)^k.$$

Therefore we have that  $\theta$  is Hölder continuous at  $(t_0, x_0)$  with exponent

$$r = \frac{\log(1 - \frac{\zeta}{4})}{\log(D)}.$$

Since  $r$  does not depend on the choice of  $(t_0, x_0) \in [T_0, T] \times \mathbb{R}^2$  and  $\theta$  is smooth up to time  $T_0$ , the lemma is complete.  $\square$

#### 4. BOOTSTRAPPING

We now show that for every  $T$ ,  $\partial_\nu \Psi_1(t, x) \in L^\infty([0, T]; \dot{B}_{\infty, \infty}^1(\mathbb{R}^2))$ , which will give that  $\bar{\nabla}^\perp \Psi_1 \in L^\infty([0, T] \times [0, \infty); \dot{B}_{\infty, \infty}^1(\mathbb{R}^2))$ . Here  $[0, T] \times [0, \infty)$  denotes  $t$  and  $z$ , and  $\mathbb{R}^2$  includes points  $x = (x_1, x_2)$  belonging to flat planes  $z = z_0$ . Due to the fact that the Poisson kernel is the fundamental solution of the equation

$$(4.1) \quad \partial_t \theta + (-\bar{\Delta})^{\frac{1}{2}} \theta = 0$$

we need the following lemma. This lemma provides an estimate on the regularity of the solution to an inhomogeneous version of Eq. (4.1). In [7], the authors used the Poisson kernel to bootstrap the regularity beyond  $C^\alpha$ ; however, due to the fact that  $\Psi_2|_{z=0} \in \dot{B}_{\infty, \infty}^2$  and the fact that  $\dot{B}_{\infty, \infty}^\alpha$  does not coincide with  $\dot{C}^\alpha$  when  $\alpha$  is an integer, we cannot use that argument.

**Lemma 4.1.** *Let  $f(t, x) \in L^\infty([0, T]; (\dot{B}_{\infty, \infty}^1(\mathbb{R}^2))^2)$ , let  $\mathcal{P}(t, x)$  be the Poisson kernel for  $x \in \mathbb{R}^2$ , and define*

$$g(t, x) = \int_0^t \int_{\mathbb{R}^2} \mathcal{P}(t-s, x-y) \operatorname{div}(f(s, y)) dy ds.$$

*Then  $g(t, x)$  belongs to  $L^\infty([0, T]; \dot{B}_{\infty, \infty}^1(\mathbb{R}^2))$ .*

*Proof.* Let us first give an intuition as to why the lemma holds. Integrating by parts and examining the scaling of the gradient of the Poisson kernel in space and time suggest that the expression should behave like a singular integral (in space-time) near  $s = t, x = y$ . The difference quotient characterization of  $\dot{B}_{\infty, \infty}^1$  suggests a potential theoretic argument (see [22], in which the authors show how the Riesz transform affects a modulus of continuity). The possible difficulty in applying this argument lies in the lack of regularity of  $f$  in time. However, the fact that the gradient of the Poisson kernel has mean value zero in  $x$  for each fixed time  $t$  makes up for this lack of regularity.

We begin by using the difference quotient characterization of  $\dot{B}_{\infty, \infty}^1$  to show that if  $f(t, x) \in L^\infty([0, T]; (\operatorname{Lip}(\mathbb{R}^2))^2)$ , then  $g(t, x) \in L^\infty([0, T]; \dot{B}_{\infty, \infty}^1(\mathbb{R}^2))$ . Then using the fact that the Littlewood-Paley projections  $\Delta_j f$  are uniformly Lipschitz in  $j$  will finish the proof.

Let us therefore proceed assuming that  $f \in L^\infty([0, T]; (\operatorname{Lip}(\mathbb{R}^2))^2)$ . We wish to show that the quantity  $|g(t, x+z) + g(t, x-z) - 2g(t, x)|$  is  $\mathcal{O}(|z|)$  uniformly in  $t$  and  $x$ . The proof proceeds by integrating by parts and splitting the integral into four terms, estimating each separately. In the subsequent calculations, we use the notation  $B((t, x), r) := \{(t, x) : \sqrt{t^2 + x^2} \leq r\}$ .

$$\begin{aligned} |g(t, x+z) + g(t, x-z) - 2g(t, x)| &= \left| \int_0^t \int_{\mathbb{R}^2} [\mathcal{P}(t-s, x+z-y) + \mathcal{P}(t-s, x-z-y) \right. \\ &\quad \left. - 2\mathcal{P}(t-s, x-y)] \operatorname{div}(f(s, y)) dy ds \right| \\ &= \left| \int_0^t \int_{\mathbb{R}^2} [\nabla_x \mathcal{P}(t-s, x+z-y) + \nabla_x \mathcal{P}(t-s, x-z-y) \right. \end{aligned}$$



$$\begin{aligned}
& \left| -2\nabla_x \mathcal{P}(t-s, x-y) \cdot (f(s, y)) dy ds \right| \\
& \leq \left| \iint_{B((t-s, x+z-y), 2|z|)} \nabla_x \mathcal{P}(t-s, x+z-y) \cdot f(s, y) dy ds \right| \\
& + \left| \iint_{B((t-s, x-z-y), 2|z|)} \nabla_x \mathcal{P}(t-s, x-z-y) \cdot f(s, y) dy ds \right| \\
& + \left| \iint_{B((t-s, x-y), 2|z|)} \nabla_x \mathcal{P}(t-s, x-y) \cdot f(s, y) dy ds \right| \\
& + \left| \iint_{B((t-s, x+z-y), 2|z|)^c} \nabla_x \mathcal{P}(t-s, x+z-y) \cdot f(s, y) dy ds \right. \\
& \quad + \iint_{B((t-s, x-z-y), 2|z|)^c} \nabla_x \mathcal{P}(t-s, x-z-y) \cdot f(s, y) dy ds \\
& \quad \left. + \iint_{B((t-s, x-y), 2|z|)^c} \nabla_x \mathcal{P}(t-s, x-y) \cdot f(s, y) dy ds \right| \\
(4.2) \quad & = I + II + III + IV.
\end{aligned}$$

The first three terms are all integrals over a ball of radius  $2|z|$  around the respective singularities of the three points  $(t, x+z)$ ,  $(t, x-z)$ , and  $(t, x)$  and are estimated in the same way. We calculate  $III$  using that the gradient of the Poisson kernel has mean value zero in space for each time, the Lipschitz regularity of  $f$ , and integration in polar coordinates (in space-time).

$$\begin{aligned}
III &= \left| \iint_{B((t-s, x-y), 2|z|)} \nabla_x \mathcal{P}(t-s, x-y) \cdot [f(s, y) - f(s, x)] dy ds \right| \\
&\leq \iint_{B((t-s, x-y), 2|z|)} \left| \frac{(t-s)(x-y)}{(|x-y|^2 + (t-s)^2)^{\frac{5}{2}}} \right| |x-y| \|f\|_{L^\infty(\text{Lip})} dy ds \\
&\leq \int_0^{2|z|} \|f\|_{L^\infty(\text{Lip})} dr \\
(4.3) \quad &\leq 2\|f\|_{L^\infty(\text{Lip})}|z|.
\end{aligned}$$

Since an identical estimate holds for  $I$  and  $II$ , it remains to estimate  $IV$ . This term contains three integrals, each integrated over the complement of a ball around the singularities corresponding to  $(t, x+z)$ ,  $(t, x-z)$ , and  $(t, x)$ . Notice that each domain of integration contains the set  $B((t-s, x-y), 3|z|)^c$ . Using again the mean value condition on the gradient of the Poisson kernel, we have that

$$\begin{aligned}
IV &= \left| \iint_{B((t-s, x+z-y), 2|z|)^c} \nabla_x \mathcal{P}(t-s, x+z-y) \cdot [f(s, y) - f(s, x)] dy ds \right. \\
& \quad + \iint_{B((t-s, x-z-y), 2|z|)^c} \nabla_x \mathcal{P}(t-s, x-z-y) \cdot [f(s, y) - f(s, x)] dy ds \\
& \quad \left. + \iint_{B((t-s, x-y), 2|z|)^c} \nabla_x \mathcal{P}(t-s, x-y) \cdot [f(s, y) - f(s, x)] dy ds \right|
\end{aligned}$$

$$\begin{aligned}
&\leq \iint_{B((t-s, x-y), 3|z|)^c} \left| [\nabla_x \mathcal{P}(t-s, x+z-y) + \nabla_x \mathcal{P}(t-s, x-z-y) \right. \\
&\quad \left. + \nabla_x \mathcal{P}(t-s, x-y)] \cdot [f(s, y) - f(s, x)] \right| dy ds \\
&\quad + \iint_{2|z| < \sqrt{|x-y|^2 + (t-s)^2} < 3|z|} \left| \nabla_x \mathcal{P}(t-s, x-y) \cdot [f(s, y) - f(s, x)] \right| dy ds \\
&\quad + \iint_{\frac{3}{2}|z| < \sqrt{|x+z-y|^2 + (t-s)^2} < 4|z|} \left| \nabla_x \mathcal{P}(t-s, x+z-y) \cdot [f(s, y) - f(s, x)] \right| dy ds \\
&\quad + \iint_{\frac{3}{2}|z| < \sqrt{|x-z-y|^2 + (t-s)^2} < 4|z|} \left| \nabla_x \mathcal{P}(t-s, x-z-y) \cdot [f(s, y) - f(s, x)] \right| dy ds \\
(4.4) \quad &= IV_1 + IV_2 + IV_3 + IV_4.
\end{aligned}$$

The estimate for  $IV_2$  is identical to  $III$ . In addition, although  $(s, x)$  is not at the center of the annulus in  $IV_3$ ,  $|x-y|$  is comparable to  $|x+z-y|$  in the annulus, and therefore  $IV_3$  can be estimated in the same manner as  $III$ . The same estimate also holds for  $IV_4$ . Thus it remains to estimate  $IV_1$ . Here the cancellation of the difference quotient condition (compared to a Lipschitz condition) is used in a crucial way. Noticing that the first order Taylor expansion (in space and time) of

$$\nabla_x \mathcal{P}(t-s, x+z-y) + \nabla_x \mathcal{P}(t-s, x-z-y) - 2\nabla_x \mathcal{P}(t-s, x-y)$$

around  $(t-s, x-y)$  vanishes, we have that

$$\begin{aligned}
&|\nabla_x \mathcal{P}(t-s, x+z-y) + \nabla_x \mathcal{P}(t-s, x-z-y) - 2\nabla_x \mathcal{P}(t-s, x-y)| \\
&\leq |\nabla^2(\nabla_x \mathcal{P}(t-s, x-y))| \cdot |z|^2 \\
(4.5) \quad &\leq \frac{|z|^2}{(|x-y|^2 + (t-s)^2)^5}.
\end{aligned}$$

Using (4.5), the Lipschitz regularity of  $f$ , and integration in polar coordinates, we have that

$$\begin{aligned}
IV_1 &= \iint_{B((t-s, x-y), 3|z|)^c} \left| [\nabla_x \mathcal{P}(t-s, x+z-y) + \nabla_x \mathcal{P}(t-s, x-z-y) \right. \\
&\quad \left. + \nabla_x \mathcal{P}(t-s, x-y)] \cdot [f(s, y) - f(s, x)] \right| dy ds \\
&\leq \iint_{B((t-s, x-y), 3|z|)^c} \frac{|z|^2 |x-y| \cdot \|f\|_{L^\infty(\text{Lip})}}{(|x-y|^2 + (t-s)^2)^5} dy ds \\
&\leq |z|^2 \|f\|_{L^\infty(\text{Lip})} \int_{3|z|}^\infty \frac{1}{r^2} dr \\
(4.6) \quad &\leq \|f\|_{L^\infty(\text{Lip})} |z|.
\end{aligned}$$

Assembling the estimates (4.3)-(4.6) into (4.2) shows that the desired quantity is  $\mathcal{O}(|z|)$ . Thus we have shown that if  $f(t, x) \in L^\infty([0, T]; (\text{Lip}(\mathbb{R}^2))^2)$ , then  $g(t, x) \in L^\infty([0, T]; \dot{B}_{\infty, \infty}^1(\mathbb{R}^2))$ . It remains to weaken the assumption to  $f(t, x) \in L^\infty([0, T]; (\dot{B}_{\infty, \infty}^1(\mathbb{R}^2))^2)$ . Since  $\Delta_j f \in L^\infty([0, T]; (\text{Lip}(\mathbb{R}^2))^2)$  with norm no larger than  $\|f\|_{L^\infty((\dot{B}_{\infty, \infty}^1)^2)}$ , we can apply the above to

$\Delta_j g(t, x)$  to conclude that  $\Delta_j g(t, x) \in L^\infty([0, T]; \dot{B}_{\infty, \infty}^1(\mathbb{R}^2))$  uniformly in  $j$ . Thus

$$\begin{aligned}
\|g(t, \cdot)\|_{\dot{B}_{\infty, \infty}^1} &= \sup_j 2^j \|\Delta_j g(t, \cdot)\|_\infty \\
&= \sup_j 2^j \|\Delta_j (\sum_k \Delta_k g(t, \cdot))\|_\infty \\
&\leq \sup_j 2^j (\|\Delta_j \Delta_{j-1} g(t, \cdot)\|_\infty + \|\Delta_j \Delta_j g(t, \cdot)\|_\infty + \|\Delta_j \Delta_{j+1} g(t, \cdot)\|_\infty) \\
&\leq C \|\Delta_j g(t, \cdot)\|_{\dot{B}_{\infty, \infty}^1} \\
&< \infty.
\end{aligned}$$

□

We can now show that the regularity of  $\partial_\nu \Psi_1$  can be bootstrapped all the way to  $\dot{B}_{\infty, \infty}^1(\mathbb{R}^2)$ . Let  $\Psi$  be a strong solution to (QG) on  $[0, T]$ . We will split up  $\partial_\nu \Psi_1$  into two pieces; one to which Lemma 4.1 can be applied, and another to which Proposition 2.7 can be applied. First, let  $f$  be the solution to

$$\begin{cases} \partial_t f + (-\bar{\Delta})^{\frac{1}{2}} f = \operatorname{div}(\bar{\nabla} \Psi_2) \\ f|_{t=0} = 0. \end{cases}$$

Then the solution  $f$  has a representation formula via the Poisson kernel. We can therefore apply Lemma 4.1 to  $f$ , and we have  $f \in L^\infty([0, T]; \dot{B}_{\infty, \infty}^1(\mathbb{R}^2))$ . Now considering  $h = \partial_\nu \Psi_1 - f$ , we see that  $h$  solves the equation

$$\partial_t h + u \cdot \bar{\nabla}(\partial_\nu \Psi_1) + (-\bar{\Delta})^{\frac{1}{2}} h = 0.$$

Here  $u = \bar{\nabla}^\perp \Psi_1 + \bar{\nabla}^\perp \Psi_2$ . By Lemma 3.6,  $\partial_\nu \Psi_1 \in L^\infty([0, T]; C^\delta(\mathbb{R}^2))$  for some  $\delta > 0$ . Therefore, applying the Riesz transforms and using Proposition 2.3,  $\bar{\nabla}^\perp \Psi_1|_{z=0} \in L^\infty([0, T]; C^\delta(\mathbb{R}^2))$  as well. Also,  $\bar{\nabla}^\perp \Psi_2|_{z=0} \in L^\infty([0, T]; C^\delta(\mathbb{R}^2))$  for every  $\delta \in (0, 1)$ . Therefore, as long as  $p$  is large enough, Proposition 2.7 applies, showing that  $h \in L^\infty([0, T]; C^{2\delta-\epsilon}(\mathbb{R}^2))$  for every  $\epsilon > 0$ . Repeating this argument a finite number of times shows that  $h \in L^\infty([0, T]; C^{1, \alpha}(\mathbb{R}^2))$  for some  $\alpha > 0$ , and in particular,  $h \in L^\infty([0, T]; \dot{B}_{\infty, \infty}^1(\mathbb{R}^2))$ . Then since  $\partial_\nu \Psi_1 = f + h$ , we have that  $\partial_\nu \Psi_1 \in L^\infty([0, T]; \dot{B}_{\infty, \infty}^1(\mathbb{R}^2))$ .

We now show that for any  $z$ ,  $\nabla \Psi_1(\cdot, z)$  enjoys the same regularity in  $x$  as  $\partial_\nu \Psi_1$ . Recalling that the  $L^1(\mathbb{R}^2)$  norm of the Poisson kernel  $\mathcal{P}_z(x) := \mathcal{P}(x, z)$  is equal to 1 for any  $z$ , we can say that for all  $j$ ,

$$\|\Delta_j (\mathcal{P}_z * (\partial_\nu \Psi_1))\|_{L^\infty(\mathbb{R}^2)} \leq \|\Delta_j (\partial_\nu \Psi_1)\|_{L^\infty(\mathbb{R}^2)}$$

(where the Littlewood-Paley projection is in  $x$  only). This shows that  $(\mathcal{P}_z * (\partial_\nu \Psi_1)) \in \dot{B}_{\infty, \infty}^1(\mathbb{R}^2)$  with norm less than or equal to that of  $\partial_\nu \Psi_1$ . Furthermore, this estimate is uniform in  $z$ . Next, we note that

$$\nabla \Psi_1(z, x) = (\mathcal{P}_z * (\partial_\nu \Psi_1))(x), \mathcal{R}_1(\mathcal{P}_z * (\partial_\nu \Psi_1))(x), \mathcal{R}_2(\mathcal{P}_z * (\partial_\nu \Psi_1))(x))$$

where  $\mathcal{R}_i$  is the  $i^{\text{th}}$  Riesz transform. Using the boundedness of the Riesz transforms on Besov spaces and the above observations regarding the Poisson kernel, we have that  $\nabla \Psi_1 \in L^\infty([0, T] \times [0, \infty); \dot{B}_{\infty, \infty}^1(\mathbb{R}^2))$ . Finally, using that  $\nabla \Psi_2 \in L^\infty([0, T] \times [0, \infty); \dot{B}_{\infty, \infty}^1(\mathbb{R}^2))$  by Proposition 2.11, we have shown the following:

**Theorem 4.2.** *Let  $\Psi$  be a strong solution to (QG) on  $[0, T]$ ; then  $\Psi$  satisfies*

$$\nabla \Psi \in L^\infty([0, T] \times [0, \infty); \dot{B}_{\infty, \infty}^1(\mathbb{R}^2)).$$

## 5. PROPAGATION OF REGULARITY

We begin by using the transport equations on both  $\nabla \Psi$  and  $\Delta \Psi$  to show that smoothness in the flat variables is propagated in time. Then, using this result in conjunction with the stratification of the flow will show that smoothness in all variables is propagated in time. Since the local existence theorem gives existence of strong solutions on a time interval which depends only on  $\|\nabla \Psi_0\|_{H^3(\mathbb{R}_+^3)}$ , obtaining a differential inequality which bounds  $\|\nabla \Psi(t)\|_{H^3(\mathbb{R}_+^3)}$  in time allows us to apply a continuation principle, thus showing that solutions are smooth for all time.

**Lemma 5.1.** *Let  $\Psi$  be a solution to (QG). Then if  $\bar{\nabla}^{s+1}(\nabla \Psi)|_{t=0}, \bar{\nabla}^s(\Delta \Psi)|_{t=0} \in L^2(\mathbb{R}_+^3)$  for some  $s \geq 2$ , then  $\bar{\nabla}^{s+1}(\nabla \Psi), \bar{\nabla}^s(\Delta \Psi) \in L^2(\mathbb{R}_+^3)$  for all times  $t > 0$ .*

*Proof.* Recall that for  $s = |\alpha|$ , Proposition 2.6 gives the commutator estimate

$$\|D^\alpha(fg) - fD^\alpha g\|_{L^2} \leq C(s) (\|\nabla f\|_{L^\infty} \|\nabla^{(s-1)} g\|_{L^2} + \|g\|_{L^\infty} \|\nabla^s f\|_{L^2}).$$

Also recall that Proposition 2.5 provides the bound

$$\|h\|_{L^\infty} \leq C\|h\|_{H^{-1}} + C\|h\|_{\dot{B}_{\infty, \infty}^0} \left(1 + \log \frac{\|h\|_{\dot{H}^{\frac{3}{2}}}}{\|h\|_{\dot{B}_{\infty, \infty}^0}}\right).$$

Finally, recall that taking a trace gives that  $u|_{z=z_0} \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^2)$  if  $\nabla u \in L^2(\mathbb{R}_+^3)$ , with the trace estimate  $\|u(z_0, \cdot)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^2)} \leq \|\nabla u\|_{L^2(\mathbb{R}_+^3)}$ . Using the fact that  $\partial_{zz}\Psi = \Delta \Psi - \bar{\Delta} \Psi$  and applying the trace estimate with  $u = \bar{\nabla}^2(\nabla \Psi)$  gives that

$$\sup_z \|\nabla \Psi(z, \cdot)\|_{\dot{H}^{\frac{5}{2}}} = \sup_z \|\bar{\nabla}^2(\nabla \Psi)(z, \cdot)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^2)} \leq \|\bar{\nabla}^3(\nabla \Psi)\|_{L^2(\mathbb{R}_+^3)} + \|\bar{\nabla}^2(\Delta \Psi)\|_{L^2(\mathbb{R}_+^3)}.$$

From Theorem 4.2,  $\nabla \Psi \in L^\infty([0, T] \times [0, \infty); \dot{B}_{\infty, \infty}^1(\mathbb{R}^2))$ . In addition, Proposition 2.11 gives that  $\|\bar{\nabla}(\nabla \Psi)(z, \cdot)\|_{H^{-1}(\mathbb{R}^2)} \leq \|\nabla \Psi\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}_+^3)}$  is uniformly bounded on finite time intervals. We apply Proposition 2.5 to  $h = \bar{\nabla}(\nabla \Psi)$ , Proposition 2.11, Theorem 4.2, and (5.1) to obtain the following:

$$\begin{aligned} \|\bar{\nabla}(\nabla \Psi)\|_{L^\infty(\mathbb{R}_+^3)} &= \sup_z \|\bar{\nabla}(\nabla \Psi)(z, \cdot)\|_{L^\infty(\mathbb{R}^2)} \\ &\leq C \sup_z \left( \|\bar{\nabla}(\nabla \Psi)(z, \cdot)\|_{H^{-1}} + \|\bar{\nabla}(\nabla \Psi)(z, \cdot)\|_{\dot{B}_{\infty, \infty}^0} \left(1 + \log \frac{\|\bar{\nabla}(\nabla \Psi)(z, \cdot)\|_{\dot{H}^{\frac{3}{2}}}}{\|\bar{\nabla}(\nabla \Psi)(z, \cdot)\|_{\dot{B}_{\infty, \infty}^0}}\right) \right) \\ &\leq C \sup_z \left( 1 + \|\bar{\nabla}(\nabla \Psi)(z, \cdot)\|_{\dot{B}_{\infty, \infty}^0} \left( \log \|\nabla \Psi(z, \cdot)\|_{\dot{H}^{\frac{5}{2}}} - \log \|\bar{\nabla}(\nabla \Psi)(z, \cdot)\|_{\dot{B}_{\infty, \infty}^0} \right) \right) \\ &\leq C \sup_z \left( 1 + \log_+ \|\nabla \Psi(z, \cdot)\|_{\dot{H}^{\frac{5}{2}}} \right) \\ &\leq C \left( 1 + \log_+ \left( \|\bar{\nabla}^3(\nabla \Psi)\|_{L^2(\mathbb{R}_+^3)} + \|\bar{\nabla}^2(\Delta \Psi)\|_{L^2(\mathbb{R}_+^3)} \right) \right). \end{aligned}$$

We shall obtain a differential inequality from the transport equations on  $\nabla\Psi$  and  $\Delta\Psi$ . Beginning with the former, we have from Proposition 2.9 that

$$\partial_t(\nabla\Psi) + \mathbb{P}_\nabla(\bar{\nabla}^\perp\Psi \cdot \bar{\nabla}(\nabla\Psi)) = \nabla F.$$

We shall apply the commutator bound by putting  $f = \bar{\nabla}^\perp\Psi$ ,  $g = \bar{\nabla}(\nabla\Psi)$ , and applying a differential operator  $\bar{D}^\alpha$  with  $|\alpha| = s + 1$ . Using (5.2) and the fact that  $|s| \geq 2$ , we have

$$\begin{aligned} \left\| [\bar{\nabla}^\perp\Psi, \bar{D}^\alpha](\bar{\nabla}(\nabla\Psi)(z, \cdot)) \right\|_{L^2(\mathbb{R}^2)} &\leq C \left( \left\| \bar{\nabla}(\bar{\nabla}^\perp\Psi)(z, \cdot) \right\|_{L^\infty} \left\| \bar{\nabla}^s(\bar{\nabla}(\nabla\Psi))(z, \cdot) \right\|_{L^2} \right. \\ &\quad \left. + \left\| \bar{\nabla}(\nabla\Psi)(z, \cdot) \right\|_{L^\infty} \left\| \bar{\nabla}^{s+1}(\bar{\nabla}^\perp\Psi)(z, \cdot) \right\|_{L^2} \right) \\ &\leq C \left\| \bar{\nabla}^{s+1}(\nabla\Psi)(z, \cdot) \right\|_{L^2} \left\| \bar{\nabla}(\nabla\Psi)(z, \cdot) \right\|_{L^\infty} \\ &\leq C \left\| \bar{\nabla}^{s+1}(\nabla\Psi)(z, \cdot) \right\|_{L^2} \left( 1 + \log_+ \left( \left\| \bar{\nabla}^3(\nabla\Psi) \right\|_{L^2(\mathbb{R}_+^3)} + \left\| \bar{\nabla}^2(\Delta\Psi) \right\|_{L^2(\mathbb{R}_+^3)} \right) \right). \end{aligned}$$

Applying the differential operator  $\bar{D}^\alpha$  with  $|\alpha| = s+1 \geq 3$ , multiplying by  $\bar{D}^\alpha\nabla\Psi$ , integrating by parts, and utilizing the commutator estimate gives

$$\begin{aligned} \frac{1}{2} \frac{\partial}{\partial t} \int_{\mathbb{R}_+^3} |\bar{D}^\alpha\nabla\Psi|^2 &= \int_{\mathbb{R}_+^3} \mathbb{P}_\nabla \left[ [\bar{\nabla}^\perp\Psi, \bar{D}^\alpha](\bar{\nabla}(\nabla\Psi)) \right] \cdot \nabla \bar{D}^\alpha\Psi + \int_{\mathbb{R}_+^3} \nabla \bar{D}^\alpha F \cdot \nabla \bar{D}^\alpha\Psi \\ &= \int_{\mathbb{R}_+^3} [\bar{\nabla}^\perp\Psi, \bar{D}^\alpha](\bar{\nabla}(\nabla\Psi)) \cdot \nabla \bar{D}^\alpha\Psi + \int_{\mathbb{R}^2} (\bar{D}^\alpha(\partial_\nu F))(\bar{D}^\alpha\Psi) \\ &= \int_{\mathbb{R}_+^3} [\bar{\nabla}^\perp\Psi, \bar{D}^\alpha](\bar{\nabla}(\nabla\Psi)) \cdot \nabla \bar{D}^\alpha\Psi + \int_{\mathbb{R}^2} (\bar{D}^\alpha(\bar{\Delta}\Psi))(\bar{D}^\alpha\Psi) \\ &\leq \int_0^\infty \int_{\mathbb{R}^2} [\bar{\nabla}^\perp\Psi(z, \cdot), \bar{D}^\alpha](\bar{\nabla}(\nabla\Psi)(z, \cdot)) \cdot \nabla \bar{D}^\alpha\Psi(z, \cdot) \, dx \, dz \\ &\leq \int_0^\infty \left\| [\bar{\nabla}^\perp\Psi(z, \cdot), \bar{D}^\alpha](\bar{\nabla}(\nabla\Psi)(z, \cdot)) \right\|_{L^2(\mathbb{R}^2)} \left\| \bar{D}^\alpha\nabla\Psi(z, \cdot) \right\|_{L^2(\mathbb{R}^2)} \, dz \\ &\leq C \int_0^\infty \left\| \bar{\nabla}^{s+1}(\nabla\Psi)(z, \cdot) \right\|_{L^2(\mathbb{R}^2)}^2 \left( 1 + \log_+ \left( \left\| \bar{\nabla}^3(\nabla\Psi) \right\|_{L^2(\mathbb{R}_+^3)} + \left\| \bar{\nabla}^2(\Delta\Psi) \right\|_{L^2(\mathbb{R}_+^3)} \right) \right) \, dz \\ &\leq C \left\| \bar{\nabla}^{s+1}(\nabla\Psi) \right\|_{L^2(\mathbb{R}_+^3)}^2 \left( 1 + \log_+ \left( \left\| \bar{\nabla}^3(\nabla\Psi) \right\|_{L^2(\mathbb{R}_+^3)} + \left\| \bar{\nabla}^2(\Delta\Psi) \right\|_{L^2(\mathbb{R}_+^3)} \right) \right). \end{aligned}$$

We now move to the transport equation on  $\Delta\Psi$ :

$$\partial_t(\Delta\Psi) + \bar{\nabla}^\perp\Psi \cdot \bar{\nabla}(\Delta\Psi) = 0.$$

We shall apply the commutator bound by putting  $f = \bar{\nabla}^\perp\Psi$ ,  $g = \Delta\Psi$ , and applying a differential operator  $\bar{D}^\alpha$  with  $|\alpha| = s$ . Using (5.2) and the fact that  $|s| \geq 2$ , we have

$$\begin{aligned} \left\| [\bar{\nabla}^\perp\Psi, \bar{D}^\alpha\bar{\nabla} \cdot](\Delta\Psi)(z, \cdot) \right\|_{L^2(\mathbb{R}^2)} &\leq C \left( \left\| \bar{\nabla}(\bar{\nabla}^\perp\Psi)(z, \cdot) \right\|_{L^\infty} \left\| \bar{\nabla}^s(\Delta\Psi)(z, \cdot) \right\|_{L^2} \right. \\ &\quad \left. + \left\| \Delta\Psi(z, \cdot) \right\|_{L^\infty} \left\| \bar{\nabla}^{s+1}(\bar{\nabla}^\perp\Psi)(z, \cdot) \right\|_{L^2} \right) \end{aligned}$$

$$\begin{aligned}
&\leq C \left( \|\bar{\nabla}^s(\Delta\Psi)(z, \cdot)\|_{L^2} \|\bar{\nabla}(\nabla\Psi)(z, \cdot)\|_{L^\infty} + \|\bar{\nabla}^{s+1}(\bar{\nabla}^\perp\Psi)(z, \cdot)\|_{L^2} \right) \\
&\leq C \left( \|\bar{\nabla}^s(\Delta\Psi)(z, \cdot)\|_{L^2} + \|\bar{\nabla}^{s+1}(\nabla\Psi)(z, \cdot)\|_{L^2} \right) \\
&\quad \times \left( 1 + \log_+ \left( \|\bar{\nabla}^3(\nabla\Psi)\|_{L^2(\mathbb{R}_+^3)} + \|\bar{\nabla}^2(\Delta\Psi)\|_{L^2(\mathbb{R}_+^3)} \right) \right).
\end{aligned}$$

Applying the differential operator  $\bar{D}^\alpha$  with  $|\alpha| = s \geq 2$ , multiplying by  $\bar{D}^\alpha \Delta\Psi$ , integrating by parts, and utilizing the commutator estimate gives

$$\begin{aligned}
\frac{1}{2} \frac{\partial}{\partial t} \int_{\mathbb{R}_+^3} |\bar{D}^\alpha \Delta\Psi|^2 &= \int_{\mathbb{R}_+^3} [\bar{\nabla}^\perp \Psi, \bar{D}^\alpha \bar{\nabla} \cdot] (\Delta\Psi) \cdot \bar{D}^\alpha \Delta\Psi \\
&= \int_0^\infty \int_{\mathbb{R}^2} [\bar{\nabla}^\perp \Psi(z, x), \bar{D}^\alpha \bar{\nabla} \cdot] (\Delta\Psi)(z, x) \cdot \bar{D}^\alpha \Delta\Psi(z, x) dx dz \\
&\leq \int_0^\infty \left\| [\bar{\nabla}^\perp \Psi(z, \cdot), \bar{D}^\alpha \bar{\nabla} \cdot] (\Delta\Psi)(z, \cdot) \right\|_{L^2(\mathbb{R}^2)} \|\bar{D}^\alpha \Delta\Psi(z, \cdot)\|_{L^2(\mathbb{R}^2)} dz \\
&\leq C \int_0^\infty \left( \|\bar{\nabla}^s(\Delta\Psi)(z, \cdot)\|_{L^2(\mathbb{R}^2)}^2 + \|\bar{\nabla}^{s+1}(\nabla\Psi)(z, \cdot)\|_{L^2(\mathbb{R}^2)} \|\bar{\nabla}^s(\Delta\Psi)(z, \cdot)\|_{L^2(\mathbb{R}^2)} \right) \\
&\quad \times \left( 1 + \log_+ \|\bar{\nabla}^3(\nabla\Psi)\|_{L^2(\mathbb{R}_+^3)} + \|\bar{\nabla}^2(\Delta\Psi)\|_{L^2(\mathbb{R}_+^3)} \right) dz \\
&\leq C \left( \|\bar{\nabla}^s(\Delta\Psi)\|_{L^2(\mathbb{R}_+^3)}^2 + \|\bar{\nabla}^{s+1}(\nabla\Psi)\|_{L^2(\mathbb{R}_+^3)} \|\bar{\nabla}^s(\Delta\Psi)\|_{L^2(\mathbb{R}_+^3)} \right) \\
&\quad \times \left( 1 + \log_+ \left( \|\bar{\nabla}^3(\Delta\Psi)\|_{L^2(\mathbb{R}_+^3)} + \|\bar{\nabla}^2(\Delta\Psi)\|_{L^2(\mathbb{R}_+^3)} \right) \right).
\end{aligned}$$

Therefore, we can sum over  $\alpha$  in both inequalities and apply Gronwall's inequality to the sum

$$\|\bar{\nabla}^{s+1}(\Delta\Psi)\|_{L^2(\mathbb{R}_+^3)} + \|\bar{\nabla}^s(\Delta\Psi)\|_{L^2(\mathbb{R}_+^3)},$$

showing that if  $\bar{\nabla}^{s+1}(\nabla\Psi)|_{t=0}, \bar{\nabla}^s(\Delta\Psi)|_{t=0} \in L^2(\mathbb{R}_+^3)$  for  $s \geq 2$ , then  $\bar{\nabla}^{s+1}(\nabla\Psi), \bar{\nabla}^s(\Delta\Psi) \in L^2(\mathbb{R}_+^3)$  for all times  $t$ .  $\square$

We now show that regularity in  $z$  can be propagated as well.

**Theorem 5.2.** *Let  $\Psi$  be a solution to (QG). Then if  $\nabla\Psi|_{t=0} \in H^s(\mathbb{R}_+^3)$  for some  $s \geq 3$ , then  $\nabla\Psi \in H^s(\mathbb{R}_+^3)$  for all times  $t > 0$ .*

*Proof.* Using Lemma 5.1, we have that  $\|\bar{\nabla}(\nabla\Psi)(t)\|_{L^\infty(\mathbb{R}_+^3)} < \infty$  (uniformly in time on finite time intervals). Also, observe that using the identity  $\partial_{zz} = \Delta - \bar{\Delta}$ , we have that

$$\|\nabla^s(\nabla\Psi)\|_{L^2} \leq C (\|\nabla^{s-1}(\Delta\Psi)\|_{L^2} + \|\bar{\nabla}^s(\nabla\Psi)\|_{L^2}).$$

By Lemma 5.1, we have that  $\|\bar{\nabla}^s(\nabla\Psi)\|_{L^2} < \infty$ . Thus the theorem will be shown if  $\Delta\Psi \in H^{s-1}$  for all time. Applying a differential operator  $D^\alpha$  with  $|\alpha| = s - 1 \geq 2$ , multiplying by  $D^\alpha \Delta\Psi$ , integrating by parts, and using the commutator estimate (in  $\mathbb{R}_+^3$ ) in conjunction with the above observations, we have

$$\begin{aligned}
\frac{1}{2} \frac{\partial}{\partial t} \int_{\mathbb{R}_+^3} |D^\alpha \Delta\Psi|^2 &= \int_{\mathbb{R}_+^3} [\bar{\nabla}^\perp \Psi, D^\alpha \bar{\nabla} \cdot] (\Delta\Psi) \cdot D^\alpha \Delta\Psi \\
&\leq C (\|\bar{\nabla}(\nabla\Psi)\|_{L^\infty} \|\nabla^{s-1}(\Delta\Psi)\|_{L^2} + \|\Delta\Psi\|_{L^\infty} \|\nabla^s(\nabla\Psi)\|_{L^2}) \|\nabla^{s-1}(\Delta\Psi)\|_{L^2}
\end{aligned}$$

$$\leq C(\|\nabla^{s-1}(\Delta\Psi)\|_{L^2}^2 + \|\nabla^{s-1}(\Delta\Psi)\|_{L^2})$$

Summing over  $\alpha$  and applying Gronwall's inequality now finishes the proof.  $\square$

*Proof of Theorem 1.1.* Theorem 5.2 gives the first part of Theorem 1.1; namely, if  $\nabla\Psi_0 \in H^s(\mathbb{R}_+^3)$  for some  $s \geq 3$ , then for all  $T > 0$ , there exists  $C(T, s)$  such that for all  $t \leq T$ ,  $\|\nabla\Psi(t, \cdot)\|_{H^s(\mathbb{R}_+^3)} \leq C(T, s)$ . To finish the proof, it remains to show uniqueness and regularity in time. Uniqueness follows from the usual energy method. Indeed, let  $\Psi_1, \Psi_2$  be two solutions with the same initial data  $\nabla\Psi_0 \in H^s(\mathbb{R}_+^3)$  for some  $s \geq 3$ . We will use the formulation of Proposition 2.9 with  $\tilde{\Psi} = \Psi_1 - \Psi_2$ ,  $\tilde{F} = F_1 - F_2$ . Considering the difference of the two equations, we have

$$\partial_t(\nabla\tilde{\Psi}) + \bar{\nabla}^\perp \tilde{\Psi} \cdot \bar{\nabla}(\nabla\Psi_2) + \bar{\nabla}^\perp \Psi_1 \cdot \bar{\nabla}(\nabla\tilde{\Psi}) = \nabla\tilde{F}.$$

Multiplying by  $\nabla\tilde{\Psi}$ , using the regularity of  $\nabla\Psi_2$ , and integrating by parts, we have

$$\begin{aligned} \frac{1}{2} \frac{\partial}{\partial t} \|\nabla\tilde{\Psi}\|_{L^2}^2 &= - \int_{\mathbb{R}_+^3} \left( \bar{\nabla}^\perp \tilde{\Psi} \cdot \bar{\nabla}(\nabla\Psi_2) + \bar{\nabla}^\perp \Psi_1 \cdot \bar{\nabla}(\nabla\tilde{\Psi}) \right) \cdot \nabla\tilde{\Psi} + \int_{\mathbb{R}_+^3} \nabla\tilde{F} \cdot \nabla\tilde{\Psi} \\ &= - \int_{\mathbb{R}_+^3} \left( \bar{\nabla}^\perp \tilde{\Psi} \cdot \bar{\nabla}(\nabla\Psi_2) \right) \cdot \nabla\tilde{\Psi} + \int_{\mathbb{R}^2} \bar{\Delta}\tilde{\Psi}\tilde{\Psi} \\ &\leq C\|\nabla\tilde{\Psi}\|_{L^2}^2. \end{aligned}$$

Since  $\nabla\tilde{\Psi}|_{t=0} = 0$ , Gronwall's inequality shows that  $\nabla\tilde{\Psi} = 0$  for all time. For the regularity in space and time, now assume that  $\Psi$  is a solution to (QG) with smooth initial data. Using the equalities

$$\begin{aligned} \partial_t(\Delta\Psi) &= -\bar{\nabla}^\perp \Psi \cdot \bar{\nabla}(\Delta\Psi) \\ \partial_t(\partial_\nu \Psi) &= -\bar{\nabla}^\perp \Psi \cdot \bar{\nabla}(\partial_\nu \Psi) + \bar{\Delta}\Psi \end{aligned}$$

and noticing that Theorem 5.2 gives that any spatial derivative of the right hand side in either equality is bounded, we have that  $\Delta\Psi$ ,  $\partial_\nu \Psi$  and all their spatial derivatives are  $C^1$  in time. Differentiating the equations in time and continuing inductively finishes the proof of Theorem 1.1.  $\square$

## 6. APPENDIX

We prove Proposition 2.5, following the proof of Proposition 2.104 in [2].

**Proposition 2.5.** *There exists a constant  $C$  such that for any  $h : \mathbb{R}^2 \rightarrow \mathbb{R}$ ,*

$$\|h\|_{L^\infty} \leq C\|h\|_{H^{-1}} + C\|h\|_{\dot{B}_{\infty,\infty}^0} \left( 1 + \log \frac{\|h\|_{\dot{H}^{\frac{3}{2}}}}{\|h\|_{\dot{B}_{\infty,\infty}^0}} \right).$$

*Proof.* Let us set  $\Theta(x) = 1 - \sum_{j=0}^\infty \Phi_j(x)$  where  $\Phi_j$  is the function associated to the  $j^{th}$  Littlewood-Paley projection. Notice that since  $\hat{\Theta}(\xi)$  is compactly supported, we have that

$$\|\Theta * h\|_{L^\infty} \leq C\|\hat{\Theta}\hat{h}\|_{L^1} \leq C\|\hat{\Theta}\hat{h}\|_{L^2} \leq C\|h\|_{H^{-1}}.$$

In addition, we have that by the characterizations of Besov spaces and Sobolev embedding, for  $\epsilon$  small enough,

$$\sup_{j \geq 0} 2^{j\epsilon} \|\Delta_j h\|_{L^\infty} \leq C\|h\|_{C^\epsilon} \leq C\|h\|_{\dot{H}^{\frac{3}{2}}}.$$

We therefore have that

$$\begin{aligned}
\|h\|_{L^\infty} &= \left\| \Theta * h + \sum_{j=0}^{\infty} \Delta_j h \right\|_{L^\infty} \\
&\leq \|\Theta * h\|_{L^\infty} + \sum_{j=0}^{N-1} \|\Delta_j h\|_{L^\infty} + \sum_{j=N}^{\infty} 2^{j\epsilon} \|\Delta_j h\|_{L^\infty} 2^{-j\epsilon} \\
&\leq C\|h\|_{H^{-1}} + N\|h\|_{\dot{B}_{\infty,\infty}^0} + C\|h\|_{\dot{H}^{\frac{3}{2}}} \frac{2^{-(N-1)\epsilon}}{2^\epsilon - 1}
\end{aligned}$$

and taking

$$N = 1 + \left( \frac{1}{\epsilon} \log_2 \frac{\|h\|_{\dot{H}^{\frac{3}{2}}}}{\|h\|_{\dot{B}_{\infty,\infty}^0}} \right)$$

finishes the proof.  $\square$

Proposition 2.104 in [2] replaces the homogeneous Besov norm (denoted without a dot as  $B_{\infty,\infty}^0$ ) with an inhomogeneous Besov norm, rendering the control on the low frequencies by the  $H^{-1}$  norm unnecessary.

Let us now use Proposition 2.104 from [2] to provide a short justification of the construction of the bump functions  $\gamma_k$  in the proof of Lemma 3.2. Let  $\gamma_k$  be a smooth bump function compactly supported in  $B_{\frac{1}{2}+2^{-k-1}}$ , equal to  $\frac{1}{2}+2^{-k-1}$  on  $B_{\frac{1}{2}+2^{-k-2}}$ , and with  $\|\bar{\nabla}\gamma_k\|_{L^\infty} \leq C2^k$ . Then

$$\begin{aligned}
\|(-\bar{\Delta})^{\frac{1}{2}}\gamma_k\|_{L^\infty} &\leq \frac{C}{\epsilon} \|(-\bar{\Delta})^{\frac{1}{2}}\gamma_k\|_{B_{\infty,\infty}^0} \left( 1 + \log \frac{\|(-\bar{\Delta})^{\frac{1}{2}}\gamma_k\|_{B_{\infty,\infty}^\epsilon}}{\|(-\bar{\Delta})^{\frac{1}{2}}\gamma_k\|_{B_{\infty,\infty}^0}} \right) \\
&\leq \frac{C}{\epsilon} \|\bar{\nabla}\gamma_k\|_{B_{\infty,\infty}^0} \left( 1 + \log \frac{\|\bar{\nabla}\gamma_k\|_{B_{\infty,\infty}^\epsilon}}{\|\bar{\nabla}\gamma_k\|_{B_{\infty,\infty}^0}} \right) \\
&\leq \frac{C}{\epsilon} \|\bar{\nabla}\gamma_k\|_{L^\infty} \left( 1 + \log \frac{\|\bar{\nabla}\gamma_k\|_{C^\epsilon}}{\|\bar{\nabla}\gamma_k\|_{L^\infty}} \right) \\
&\leq \frac{C}{\epsilon} 2^k (1 + \log C2^{k(1+\epsilon)}) \\
&\leq Ck2^k.
\end{aligned}$$

Next we provide the details of the paraproduct estimate required to bootstrap the regularity past  $C^\alpha$ . The proof follows precisely the estimates of Constantin and Wu [14].

**Proposition 2.7.** *Let  $\theta$  be a Leray-Hopf weak solution to*

$$\partial_t \theta + u \cdot \bar{\nabla} \theta + (-\bar{\Delta})^{\frac{1}{2}} \theta = 0.$$

*Then if  $\operatorname{div} u = 0$  and*

$$u, \theta \in L^\infty([t_0, t]; C^\delta \cap L^2),$$

*we have*

$$\theta \in L^\infty([t_0, t]; C^{2\delta-\epsilon})$$

*for every  $\epsilon > 0$ .*



*Proof.* We follow the estimates in [14], making the appropriate minor adjustments due to the slightly different structure of the drift term and the fact that the equation is no longer supercritical. First, by interpolation and the *a priori* estimates on  $\theta$ , we have for  $\delta_1 = \delta(1 - \frac{2}{p})$  and  $p > 2$ ,

$$\begin{aligned} \|\theta\|_{\dot{B}_{p,\infty}^{\delta_1}} &= \sup_j 2^{\delta_1 j} \|\Delta_j \theta\|_{L^p} \\ &\leq \sup_j 2^{\delta(1-\frac{2}{p})j} \|\Delta_j \theta\|_{L^\infty}^{1-\frac{2}{p}} \|\Delta_j \theta\|_{L^2}^{\frac{2}{p}} \\ &\leq \|\theta\|_{C^\delta}^{1-\frac{2}{p}} \|\theta\|_{L^2}^{\frac{2}{p}} \end{aligned}$$

and thus  $\theta \in L^\infty(\dot{B}_{p,\infty}^{\delta_1})$ . The same holds for  $u$  as well. The main estimate 6.1 will show that in fact  $\theta \in L^\infty(\dot{B}_{p,\infty}^{2\delta_1} \cap C^{\delta'})$  for some  $\delta'$  to be chosen later.

We work towards a differential inequality on the Littlewood-Paley projections of  $\theta$ . Throughout the proof, we shall use the notation

$$S_j f = \sum_{k < j} \Delta_k f$$

and the following facts: for  $|j - k| \geq 2$ ,  $\Delta_j \Delta_k = 0$ , and for  $|j - k| \geq 3$ ,  $\Delta_j(S_{k-1} f \Delta_k f) = 0$ . We fix  $j \in \mathbb{Z}$  until taking the supremum at the end of the argument; specifically, no sums vary over  $j$ . Applying  $\Delta_j$  to the equation, and using Bony's paraproduct decomposition on the nonlinear term, we obtain

$$\begin{aligned} \partial_t \Delta_j \theta + (-\bar{\Delta})^{\frac{1}{2}} \Delta_j \theta &= - \sum_{|j-k| \leq 2} \Delta_j(S_{k-1} u \cdot \bar{\nabla} \Delta_k \theta) - \sum_{|j-k| \leq 2} \Delta_j(\Delta_k u \cdot \bar{\nabla} S_{k-1} \theta) \\ &\quad - \sum_{k \geq j-1} \sum_{|k-l| \leq 1} \Delta_j(\Delta_k u \cdot \nabla \Delta_l \theta). \end{aligned}$$

Multiplying by  $p|\Delta_j \theta|^{p-2} \Delta_j \theta$ , integrating in space, and applying the lower bound (refer to Chen, Miao, Zhang [10] or Wu [28])

$$\int_{\mathbb{R}^2} |\Delta_j f|^{p-2} \Delta_j f (-\bar{\Delta})^{\frac{1}{2}} \Delta_j f \, dx \geq C(p) 2^j \|\Delta_j f\|_p^p$$

we obtain

$$\frac{d}{dt} \|\Delta_j \theta\|_{L^p}^p + C 2^j \|\Delta_j \theta\|_{L^p}^p \leq I_1 + I_2 + I_3$$

with  $I_1$ ,  $I_2$ , and  $I_3$  given by

$$\begin{aligned} I_1 &= -p \sum_{|j-k| \leq 2} \int |\Delta_j \theta|^{p-2} \Delta_j \theta \cdot \Delta_j(S_{k-1} u \cdot \bar{\nabla} \Delta_k \theta) \, dx \\ I_2 &= -p \sum_{|j-k| \leq 2} \int |\Delta_j \theta|^{p-2} \Delta_j \theta \cdot \Delta_j(\Delta_k u \cdot \bar{\nabla} S_{k-1} \theta) \, dx \\ I_3 &= -p \sum_{k \geq j-1} \int |\Delta_j \theta|^{p-2} \Delta_j \theta \cdot \sum_{|k-l| \leq 1} \Delta_j(\Delta_k u \cdot \nabla \Delta_l \theta) \, dx. \end{aligned}$$

Beginning with  $I_2$ , we can apply Bernstein's inequality and Hölder's inequality to give

$$\begin{aligned}
I_2 &\leq C \|\Delta_j \theta\|_{L^p}^{p-1} \sum_{|j-k| \leq 2} \|\Delta_k u\|_{L^p} \|\nabla S_{k-1} \theta\|_{L^\infty} \\
&\leq C \|\Delta_j \theta\|_{L^p}^{p-1} \sum_{|j-k| \leq 2} \|\Delta_k u\|_{L^p} \sum_{m \leq k-1} 2^m \|\Delta_m \theta\|_{L^\infty} \\
&\leq C \|\Delta_j \theta\|_{L^p}^{p-1} \sum_{|j-k| \leq 2} \|\Delta_k u\|_{L^p} 2^{(1-\delta_1)k} \sum_{m \leq k-1} 2^{(m-k)(1-\delta_1)} 2^{m\delta_1} \|\Delta_m \theta\|_{L^\infty} \\
&\leq C \|\Delta_j \theta\|_{L^p}^{p-1} \|\theta\|_{C^{\delta_1}} \sum_{|j-k| \leq 2} \|\Delta_k u\|_{L^p} 2^{(1-\delta_1)k}.
\end{aligned}$$

as long as  $1 - \delta_1 > 0$ .

We move now to  $I_3$ . Applying Hölder and Bernstein, we have

$$\begin{aligned}
|I_3| &\leq p \|\Delta_j \theta\|_{L^p}^{p-1} \|\Delta_j \nabla \cdot \left( \sum_{k \geq j-1} \sum_{|l-k| \leq 1} \Delta_l u \Delta_k \theta \right)\|_{L^p} \\
&\leq p \|\Delta_j \theta\|_{L^p}^{p-1} 2^j \|u\|_{C^{\delta_1}} \sum_{k \geq j-1} 2^{-\delta_1 k} \|\Delta_k \theta\|_{L^p}.
\end{aligned}$$

We now bound  $I_1$ . We decompose  $I_1$  into three terms. Denoting the commutator with brackets, namely

$$[\Delta_j, S_{k-1} u \cdot \nabla] \Delta_k \theta = \Delta_j (S_{k-1} u \cdot \nabla \Delta_k \theta) - S_{k-1} u \cdot \nabla \Delta_j \Delta_k \theta$$

and using the fact that  $\sum_{|k-j| \leq 2} \Delta_k \Delta_j \theta = \Delta_j \theta$ , we can write

$$\begin{aligned}
I_1 &= -p \sum_{|j-k| \leq 2} \int |\Delta_j \theta|^{p-2} \Delta_j \theta \cdot [\Delta_j, S_{k-1} u \cdot \nabla] \Delta_k \theta \, dx \\
&\quad - p \int |\Delta_j \theta|^{p-2} \Delta_j \theta \cdot (S_j u \cdot \nabla \Delta_j \theta) \, dx \\
&\quad - p \sum_{|j-k| \leq 2} \int |\Delta_j \theta|^{p-2} \Delta_j \theta \cdot (S_{k-1} u - S_j u) \cdot \Delta_j \Delta_k \theta \, dx \\
&= I_{11} + I_{12} + I_{13}.
\end{aligned}$$

$I_{12}$  is zero because  $u$  is divergence free (for the case when  $u$  is not divergence free, see [14]). Applying Hölder and Bernstein to  $I_{13}$ , we obtain

$$\begin{aligned}
|I_{13}| &\leq p \|\Delta_j \theta\|_{L^p}^{p-1} \sum_{|j-k| \leq 2} \|S_{k-1} u - S_j u\|_{L^p} \|\nabla \Delta_j \theta\|_{L^\infty} \\
&\leq Cp \|\Delta_j \theta\|_{L^p}^{p-1} 2^{(1-\delta_1)j} \|\theta\|_{C^{\delta_1}} \sum_{|j-k| \leq 2} \|\Delta_k u\|_{L^p}.
\end{aligned}$$

We must now bound  $I_{11}$ . By Hölder's inequality,

$$|I_{11}| \leq p \|\Delta \theta\|_{L^p}^{p-1} \sum_{|j-k| \leq 2} \|[\Delta_j, S_{k-1} u \cdot \nabla] \Delta_k \theta\|_{L^p}.$$

Writing out the commutator explicitly and using the regularity on  $u$  and  $\theta$ , we have

$$[\Delta_j, S_{k-1}u \cdot \nabla] \Delta_k \theta = \int \Phi_j(x-y)(S_{k-1}(u)(x) - S_{k-1}(u)(y)) \cdot \nabla \Delta_k \theta(y) dy$$

and

$$\|S_{k-1}(u)(x) - S_{k-1}(u)(y)\|_{L^\infty} \leq \|u\|_{C^{\delta_1}} |x-y|^{\delta_1}.$$

Therefore,

$$\|[\Delta_j, S_{k-1}u \cdot \nabla] \Delta_k \theta\|_{L^p} \leq 2^{-\delta_1 j} \|u\|_{C^{\delta_1}} 2^k \|\Delta_k \theta\|_{L^p}$$

and thus

$$|I_{11}| \leq Cp \|\Delta_j \theta\|_{L^p}^{p-1} 2^{-\delta_1 j} \|u\|_{C^{\delta_1}} \sum_{|j-k| \leq 2} 2^k \|\Delta_k \theta\|_{L^p}.$$

Combining the estimates for  $I_1$ ,  $I_2$ , and  $I_3$  and eliminating the extra factors of  $\|\Delta_j \theta\|_{L^p}$ , we have

$$\begin{aligned} \frac{d}{dt} \|\Delta_j \theta\|_{L^p} + C 2^j \|\Delta_j \theta\|_{L^p} &\leq C 2^{(1-2\delta_1)j} \|\theta\|_{\dot{B}_{p,\infty}^{\delta_1}} \|u\|_{C^{\delta_1}} \\ &\quad + C 2^{-\delta_1 j} \|u\|_{C^{\delta_1}} \sum_{|j-k| \leq 2} 2^k \|\Delta_k \theta\|_{L^p} \\ &\quad + C \|\theta\|_{C^{\delta_1}} \sum_{|j-k| \leq 2} \|\Delta_k u\|_{L^p} 2^{(1-\delta_1)k} \\ &\quad + C 2^{(1-\delta_1)j} \|\theta\|_{C^{\delta_1}} \sum_{|j-k| \leq 2} \|\Delta_k u\|_{L^p} \\ &\quad + C 2^j \|u\|_{C^{\delta_1}} \sum_{k \geq j-1} 2^{-\delta_1 k} \|\Delta_k \theta\|_{L^p}. \end{aligned}$$

We can refine the bounds on the right hand side:

$$\begin{aligned} C 2^{-\delta_1 j} \|u\|_{C^{\delta_1}} \sum_{|j-k| \leq 2} 2^k \|\Delta_k \theta\|_{L^p} &= C 2^{(1-2\delta_1)j} \|u\|_{C^{\delta_1}} \sum_{|j-k| \leq 2} 2^{\delta_1 k} \|\Delta_k \theta\|_{L^p} 2^{(k-j)(1-\delta_1)} \\ &\leq C 2^{(1-2\delta_1)j} \|u\|_{C^{\delta_1}} \|\theta\|_{\dot{B}_{p,\infty}^{\delta_1}} \\ C \|\theta\|_{C^{\delta_1}} \sum_{|j-k| \leq 2} \|\Delta_k u\|_{L^p} 2^{(1-\delta_1)k} &= C 2^{(1-2\delta_1)j} \|\theta\|_{C^{\delta_1}} \sum_{|j-k| \leq 2} 2^{\delta_1 k} \|\Delta_k u\|_{L^p} 2^{(k-j)(1-2\delta_1)} \\ &\leq C 2^{(1-2\delta_1)j} \|\theta\|_{C^{\delta_1}} \|u\|_{\dot{B}_{p,\infty}^{\delta_1}} \\ C 2^{(1-2\delta_1)j} \|\theta\|_{C^{\delta_1}} \sum_{|j-k| \leq 2} \|\Delta_k u\|_{L^p} &= C 2^{(1-2\delta_1)j} \|\theta\|_{C^{\delta_1}} \sum_{|j-k| \leq 2} 2^{\delta_1 k} \|\Delta_k u\|_{L^p} 2^{(j-k)\delta_1} \\ &\leq C 2^{(1-2\delta_1)j} \|\theta\|_{C^{\delta_1}} \|u\|_{\dot{B}_{p,\infty}^{\delta_1}} \\ C 2^j \|u\|_{C^{\delta_1}} \sum_{k \geq j-1} \|\Delta_k \theta\|_{L^p} 2^{-\delta_1 k} &= C 2^{(1-2\delta_1)j} \|u\|_{C^{\delta_1}} \sum_{k \geq j-1} 2^{-2\delta_1(k-j)} 2^{\delta_1 k} \|\Delta_k \theta\|_{L^p} \\ &\leq C 2^{(1-2\delta_1)j} \|u\|_{C^{\delta_1}} \|\theta\|_{\dot{B}_{p,\infty}^{\delta_1}}. \end{aligned}$$

We can now write the differential inequality as

$$\|\Delta_j \theta(t)\|_{L^p} \leq e^{-C 2^j(t-t_0)} \|\Delta_j \theta(t_0)\|_{L^p}$$

$$+ C \int_{t_0}^t e^{-C2^j(t-s)} 2^{(1-2\delta_1)j} (||\theta||_{C^{\delta_1}} ||u||_{\dot{B}_{p,\infty}^{\delta_1}} + ||u||_{C^{\delta_1}} ||\theta||_{\dot{B}_{p,\infty}^{\delta_1}}) ds.$$

Multiplying both sides by  $2^{2\delta_1 j}$ , taking the supremum in  $j$ , and using the estimates on  $u$  and  $\theta$  obtained from interpolation, we have

$$(6.1) \quad ||\theta(t)||_{\dot{B}_{p,\infty}^{2\delta_1}} \leq C ||\theta(t_0)||_{\dot{B}_{p,\infty}^{\delta_1}} + C \max_{s \in [t_0, t]} \left( ||\theta(s)||_{C^{\delta_1}} ||u(s)||_{\dot{B}_{p,\infty}^{\delta_1}} + ||u(s)||_{C^{\delta_1}} ||\theta(s)||_{\dot{B}_{p,\infty}^{\delta_1}} \right),$$

thus proving the main estimate. Now using the Besov embedding

$$\dot{B}_{p,\infty}^{2\delta_1} \subset \dot{B}_{\infty,\infty}^{\delta'}$$

with

$$\delta' = 2\delta_1 - \frac{2}{p} = 2\delta - \frac{4\delta}{p} - \frac{4}{p}$$

and taking  $p$  to be large enough so that  $\delta' > \delta$  finishes the proof. □

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## REFERENCES

- [1] J.-P. Aubin. Un théorème de compacité. *C. R. Acad. Sci. Paris*, 256:5042–5044, 1963.
- [2] H. Bahouri, J. Chemin, and R. Danchin. *Fourier Analysis and Nonlinear Partial Differential Equations*. Springer, 2011.
- [3] J.T. Beale, T. Kato, and A. Majda. Remarks on the breakdown of smooth solutions for the 3-D Euler equations. *Comm. Math. Phys.*, 94(1):61–66, 1984.
- [4] J. Bourgain, H. Brezis, and P. Mironescu. Lifting in Sobolev spaces. *J. Anal. Math.*, 80(1):37–86, Dec 2000.
- [5] A.J. Bourgeois and J.T. Beale. Validity of the quasigeostrophic model for large-scale flow in the atmosphere and ocean. *SIAM Journal on Mathematical Analysis*, 25(4):1023–1068, 1994.
- [6] L. Caffarelli, C.H. Chan, and A. Vasseur. Regularity theory for parabolic nonlinear integral operators. *J. Amer. Math. Soc.*, 24(3):849–869, Sep 2011.
- [7] L. Caffarelli and A. Vasseur. Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation. *Ann. Math.*, 171(3):1903–1930, Apr 2010.
- [8] J.-Y. Chemin. Système primitif de l’océan-atmosphère et limite quasi-géostrophique. In *Séminaire sur les Équations aux Dérivées Partielles, 1995–1996*, Sémin. Équ. Dériv. Partielles, pages Exp. No. VII, 15. École Polytech., Palaiseau, 1996.
- [9] J.-Y. Chemin. *Fluides Parfaits Incompressibles*. Oxford University Press, 1998.
- [10] Q. Chen, C. Miao, and Z. Zhang. A new Bernstein’s inequality and the 2D dissipative quasi-geostrophic equation. *Communications in Mathematical Physics*, 271(3):821–838, 2007.
- [11] P. Constantin. *Euler Equations, Navier-Stokes Equations and Turbulence*, pages 1–43. Springer Berlin Heidelberg, Berlin, Heidelberg, 2006.
- [12] P. Constantin, A.J. Majda, and E. Tabak. *Nonlinearity*, 7(6):1495–1533, Nov 1994.
- [13] P. Constantin and V. Vicol. Nonlinear maximum principles for dissipative linear nonlocal operators and applications. *Geometric and Functional Analysis*, 22(5):1289–1321, 2012.
- [14] P. Constantin and J. Wu. Regularity of Hölder continuous solutions of the supercritical quasi-geostrophic equation. *Annales de l’Institut Henri Poincaré (C) Non Linear Analysis*, 25(6):1103–1110, 2008.
- [15] A. Córdoba and D. Córdoba. A maximum principle applied to quasi-geostrophic equations. *Communications in Mathematical Physics*, 249(3):511–528, 2004.

- [16] B. Desjardins and E. Grenier. Derivation of quasi-geostrophic potential vorticity equations. *Adv. Differential Equations*, 3(5):715–752, 1998.
- [17] H. Dong and N. Pavlović. Regularity criteria for the dissipative quasi-geostrophic equations in Hölder spaces. *Comm. Math. Phys.*, pages 801–812.
- [18] L. Grafakos. *Modern Fourier Analysis*. Springer, 2nd edition, 2009.
- [19] I.M. Held, R.T. Pierrehumbert, S.T. Garner, and K.L. Swanson. Surface quasi-geostrophic dynamics. *J. Fluid Mech.*, 282:1–20, 1995.
- [20] T. Kato. Quasi-linear equations of evolution, with applications to partial differential equations. In *Lecture Notes in Mathematics*, pages 25–70. Springer Nature, 1975.
- [21] A. Kiselev and F. Nazarov. Variation on a theme of Caffarelli and Vasseur. *Journal of Mathematical Sciences*, 166(1):31–39, Mar 2010.
- [22] A. Kiselev, F. Nazarov, and A. Volberg. *Inventiones mathematicae*, 167(3):445–453, 2007.
- [23] S. Klainerman and A. Majda. Singular limits of quasilinear hyperbolic systems with large parameters and the incompressible limit of compressible fluids. *Communications on Pure and Applied Mathematics*, 34(4):481–524, Jul 1981.
- [24] J. Pedlosky. *Geophysical Fluid Dynamics*. Springer, 1987.
- [25] M. Puel and A. Vasseur. Global weak solutions to the inviscid 3D quasi-geostrophic equation. *Communications in Mathematical Physics*, 339(3):1063–1082, 2015.
- [26] E. Stein. *Singular Integrals and Differentiability Properties of Functions*. Princeton University Press, 1971.
- [27] A. Vasseur. The De Giorgi method for elliptic and parabolic equations and some applications. *To appear in Lectures on the Analysis of Nonlinear Partial Differential Equations Vol. 4*.
- [28] J. Wu. Global solutions of the 2D dissipative quasi-geostrophic equation in Besov spaces. *SIAM Journal on Mathematical Analysis*, 36(3):1014–1030, Jan 2005.

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